Solutions to “Rings of Continuous Functions”

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Contents

1 Functions on a topological space **3**
1A Continuity on subsets . . . 3
Problem . . . 3
Solution . . . 3
1B Components of X . . . 3
Problem . . . 3
Solution . . . 3
1C C and C_b for various subspaces of $\mathbb{R}$ . . . 4
Problem . . . 4
Solution . . . 4
1D Divisors of functions . . . 6
Problem . . . 6
Solution . . . 6
1E Units . . . 6
Problem . . . 6
Solution . . . 6
1F C-embedding . . . 7
Problem . . . 7
Solution . . . 7
1G Pseudocompact spaces . . . 7
Problem . . . 7
Solution . . . 7
1H Basically and extremally disconnected spaces . . . 8
Problem . . . 8
Solution . . . 8
1I Algebra homomorphisms . . . 9
Problem . . . 9
Solution . . . 9
1J Preservation or reduction of norm . . . 9
Problem . . . 9
Solution . . . 9

2 Ideals and z-filters **11**
2A Bounded functions in ideals . . . 11
Problem . . . 11
Solution . . . 11
2B Prime ideals . . . 11
Problem . . . 11
Solution . . . 11
2C Functions congruent to constants . . . 11
Problem . . . 11
Solution . . . 11
2D z-ideals . . . 12
Problem . . . 12
Solution . . . 12
2E Prime z-filters . . . 12
Problem . . . 12
Solution . . . 12
2F Finite spaces . . . 12
Problem . . . 12

Solution 13
2G Prime vs. z-ideals in $C(\mathbb{R})$ 13
Problem 13
Solution 13
2H The identity function $\mathfrak{i}$ in $C(\mathbb{R})$ 13
Problem 13
Solution 13
2I $C(\mathbb{Q})$ and $C_b(\mathbb{Q})$ 14
Problem 14
Solution 14
2J Ideal chains in $C(\mathbb{R})$, $C(\mathbb{Q})$ and $C(\mathbb{N})$ 14
Problem 14
Solution 14
2K z-filters and C_b 15
Problem 15
Solution 15
2L e-filters and e-ideals 15
Problem 15
Solution 16
2M The uniform norm topology on C_b 19
Problem 19
Solution 19
2N The m-topology on C 20
Problem 20
Solution 20

3 Completely regular spaces **22**
3A Zero-divisors, units, square roots 22
Problem 22
Solution 22
3B Countable sets 22
Problem 22
Solution 22
3C G_δ-points of a completely regular space 23
Problem 23
Solution 23
3D Normal spaces 23
Problem 23
Solution 24
3E Nonnormal spaces 25
Problem 25
Solution 25
3F T_0-spaces 25
Problem 25
Solution 25
3G Weak topology 27
Problem 27
Solution 27
3H Completely regular family 28
Problem 28
Solution 28
3I Theorem 3.9 28
Problem 28
Solution 28

1 Functions on a topological space

1A Continuity on subsets

Problem

Let $f : X \to \mathbb{R}$ be a set mapping.

1. If the restriction of f to each of a finite number of closed sets, whose union is X, is continuous, then f is continuous.
2. If the restriction of f to each of an arbitrary number of open sets, whose union is X, is continuous, then f is continuous.
3. Let $\mathscr{S}$ be a family of closed sets whose union is X and such that $\mathscr{S}$ is locally finite. If the restriction of f to each member of $\mathscr{S}$ is continuous, then f is continuous.

Solution

1. Assume that $A_1, \dots, A_n \subseteq X$ are closed such that $f|_{A_1}, \dots, f|_{A_n}$ are continuous and $\bigcup_{i=1}^n A_i = X$. We want to show that f is continuous.

   Take any closed $B \subseteq \mathbb{R}$. Then for each $1 \leq i \leq n$, $(f|_{A_i})^{\leftarrow}[B]$ is closed in A_i. Since A_i is closed, $(f|_{A_i})^{\leftarrow}[B]$ is also closed in X. Hence we have that $f^{\leftarrow}[B] = \bigcup_{i=1}^n (f|_{A_i})^{\leftarrow}[B]$ is a finite union of closed sets and hence also closed. This shows that f is continuous. □

2. Assume that $\mathscr{A}$ is a collection of open sets such that $f|_A$, $A \in \mathscr{A}$, are continuous and $\bigcup \mathscr{A} = X$. We want to show that f is continuous.

   Take any open $B \subseteq \mathbb{R}$. Then for each $A \in \mathscr{A}$, $(f|_A)^{\leftarrow}[B]$ is open in A. Since A is open, $(f|_A)^{\leftarrow}[B]$ is also open in X. Hence we have that $f^{\leftarrow}[B] = \bigcup_{A \in \mathscr{A}} (f|_A)^{\leftarrow}[B]$ is a union of open sets and hence also open. This shows that f is continuous. □

3. Assume that $\mathscr{S}$ is a locally finite family of closed sets such that $\bigcup \mathscr{S} = X$ and each $f|_A$, $A \in \mathscr{S}$, is continuous. We want to show that f is continuous.

   Let $x \in X$ and take an open set $B \subseteq \mathbb{R}$ with $f(x) \in B$. Let $x \in C \subseteq X$ be an open neighbourhood of x such that there are only finitely many $\{A_i\}_{i=1}^n$ with $A_i \cap C \neq \emptyset$. Now define $D = C \cap (X \setminus \bigcup_{i=1}^n (f|_{A_i})^{\leftarrow}[X \setminus B])$. Then, $X \setminus B$ is closed, hence each $(f|_{A_i})^{\leftarrow}[X \setminus B]$ is closed in A_i and also in X, therefore $\bigcup_{i=1}^n (f|_{A_i})^{\leftarrow}[X \setminus B]$ is closed in X and hence $X \setminus \bigcup_{i=1}^n (f|_{A_i})^{\leftarrow}[X \setminus B]$ is open in X. So we see that D is an open set and it is also clear that $x \in D$. Now, we have that for any $d \in D$, $\forall 1 \leq i \leq n : d \notin A_i$ or $f(d) \in B$. Since $d \in C$, we must have some $1 \leq i \leq n$ with $d \in A_i$ and this shows that $f(d) \in B$. Hence we have $f[D] \subseteq B$. So we have found some open set D with $x \in D$ and $f[D] \subseteq B$. This shows that f is continuous. □

1B Components of X

Problem

1. In $C(X)$ and $C^*(X)$, all positive units have the same number of square roots.
2. X is connected if and only if $\mathbb{1}$ has exactly two square roots.
3. For finite $\mathfrak{m}$, X has $\mathfrak{m}$ components if and only if $\mathbb{1}$ has $2^{\mathfrak{m}}$ square roots. For infinite $\mathfrak{m}$, the statement is false.
4. X is connected if and only if $\mathbb{0}$ and $\mathbb{1}$ are the only idempotents in $C(X)$.
5. If X is connected, then $C(X)$ is not a direct sum of any two nontrivial rings.
6. If X is the union of disjoint nonempty open sets A and B, then $C(X)$ is isomorphic to the direct sum of $C(A)$ and $C(B)$.

Solution

1. Let $f, g \in C(X)$ be two positive units. Let $A_f = \{h \in C(X) \mid h^2 = f\}$ and $A_g = \{h \in C(X) \mid h^2 = g\}$. Then the mapping $A_f \to A_g, h \mapsto h \cdot (f^{-1} \cdot g)^{\frac{1}{2}}$ is a bijection since it has an obvious inverse and hence $|A_f| = |A_g|$. □
2. $\Longrightarrow$ Assume that X is connected. Let $f \in C(X)$ with $f^2 = \mathbb{1}$. Then f can only take the values 1 and -1 and hence must be locally constant. Since X is connected, f is either $\mathbb{0}$ or $\mathbb{1}$.

$\Longleftarrow$ By contraposition. Assume that X is disconnected and can be written as $X = A \cup B$ with $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$ and A, B open. Then we have that all of $\pm \mathbb{1}_A \pm \mathbb{1}_B$ are distinct square roots of $\mathbb{1}$, hence there are at least 4 of them. $\square$

3. First of all, assume that $\mathfrak{m}$ is finite.

   Assume that $X = \bigsqcup_{i=1}^{\mathfrak{m}} A_i$, where the A_i are the connected components of X. Then the mapping $\{1, -1\}^{\mathfrak{m}} \to C(X), (q_i)_{i=1}^{\mathfrak{m}} \mapsto \sum_{i=1}^{\mathfrak{m}} q_i \mathbb{1}_{A_i}$ is injective, so we indeed have $2^{\mathfrak{m}}$ distinct square roots. On the other hand, if f is any function only taking the values 1 and -1, it must be constant on the connected components, hence it must be of the form $\sum_{i=1}^{\mathfrak{m}} q_i \mathbb{1}_{A_i}$. Hence a space consisting of $\mathfrak{m}$ connected components has precisely $2^{\mathfrak{m}}$ square roots of $\mathbb{1}$.

   Now, take $X = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \cup \{0\} \subseteq \mathbb{R}$. Then the square roots of $\mathbb{1}$ are eventually either constantly 1 or -1. Hence there are only countably many of them. But there is no cardinal $\mathfrak{m}$ with $\aleph_0 = 2^{\mathfrak{m}}$, so the statement must fail here. $\square$

4. $\Longrightarrow$ Assume that X is connected. Let $f \in C(X)$ with $f^2 = f$. Then f can only take the values 0 and 1 and hence must be locally constant. Since X is connected, f is either $\mathbb{0}$ or $\mathbb{1}$.

   $\Longleftarrow$ By contraposition. Assume that X is disconnected and can be written as $X = A \cup B$ with $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$ and A, B open. Then we have that all of $\mathbb{0}$, $\mathbb{1}_A$, $\mathbb{1}_B$, $\mathbb{1}$ are distinct idempotents, hence there are at least 4 of them. $\square$

5. By contraposition. If $C(X)$ is a direct sum of two nontrivial rings, there exist at least 4 idempotents $0 \oplus 0$, $0 \oplus 1$, $1 \oplus 0$ and $1 \oplus 1$. So by (4.), X must be disconnected. $\square$

6. The map $C(A) \oplus C(B) \to C(X), f \oplus g \mapsto f + g$ is clearly a ring homomorphism with an inverse given by $C(X) \to C(A) \oplus C(B), f \mapsto f|_A \oplus f|_B$. Hence $C(X) \cong C(A) \oplus C(B)$. $\square$

1C C and C_b for various subspaces of $\mathbb{R}$

Problem

Consider the subspaces $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{N}$, and $N^* = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \cup \{0\}$ of $\mathbb{R}$, and the rings C and C_b for each of these spaces. Each of these rings is of cardinality $\mathfrak{c}$.

1. For each $\mathfrak{m} \leq \aleph_0$, each ring on $\mathbb{R}$, $\mathbb{N}$ or N^* contains a function having exactly $2^{\mathfrak{m}}$ square roots. If a member of $C(\mathbb{Q})$ has more than one square root, it has $\mathfrak{c}$ of them.
2. $C(\mathbb{R})$ has just two idempotents, $C(N^*)$ has exactly $\aleph_0$, and $C(\mathbb{Q})$ and $C(\mathbb{N})$ have $\mathfrak{c}$.
3. Every nonzero idempotent in $C(\mathbb{Q})$ is a sum of two nonzero idempotents. In $C(\mathbb{N})$, and in $C(N^*)$, some, but not all idempotents have this property.
4. Except for the obvious identity $C(N^*) = C_b(N^*)$, no two of the rings in question are isomorphic.
5. Each of $C(\mathbb{Q})$ and $C(\mathbb{N})$ is isomorphic with a direct sum of two copies of itself. $C(N^*)$ is isomorphic with a direct sum of two subrings, just one of which is isomorphic with $C(N^*)$.
6. The ring $C(\mathbb{R})$ is isomorphic with a proper subring. But $C(\mathbb{R})$ has no proper summand.

Solution

1. • On $\mathbb{R}$, if $\mathfrak{m}$ is finite, take

   $$f : \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} |\sin(\pi x)| & 0 \leq x \leq \mathfrak{m} \\ 0 & \text{otherwise} \end{cases}$$

   Then we have $f \in C(\mathbb{R}), C_b(\mathbb{R})$ and f clearly has $2^{\mathfrak{m}}$ square roots.

   For $\mathfrak{m} = \aleph_0$, take

   $$f : \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} |\sin(\pi x)| & 0 \leq x \\ 0 & \text{otherwise} \end{cases}$$

   Then we have $f \in C(\mathbb{R}), C_b(\mathbb{R})$ and f clearly has $\mathfrak{c}$ square roots.

   • On $\mathbb{N}$, if $\mathfrak{m}$ is finite, take

   $$f : \mathbb{N} \to \mathbb{R}, f(n) = \begin{cases} 1 & n \leq \mathfrak{m} \\ 0 & \text{otherwise} \end{cases}$$

Then we have $f \in C(\mathbb{N}), C_b(\mathbb{N})$ and f clearly has $2^{\mathfrak{m}}$ square roots.

For $\mathfrak{m} = \aleph_0$, take $\mathbb{1}$. Then we have $\mathbb{1} \in C(\mathbb{N}), C_b(\mathbb{N})$ and $\mathbb{1}$ clearly has $\mathfrak{c}$ square roots.

- On N^*, if $\mathfrak{m}$ is finite, take

$$\begin{aligned} &f : N^* \to \mathbb{R}, \\ &f(\tfrac{1}{n}) = \begin{cases} 1 & n \leq \mathfrak{m} \\ 0 & \text{otherwise} \end{cases} \\ &f(0) = 0 \end{aligned}$$

Then we have $f \in C(N^*), C_b(N^*)$ and f clearly has $2^{\mathfrak{m}}$ square roots.

For $\mathfrak{m} = \aleph_0$, take

$$\begin{aligned} &f : N^* \to \mathbb{R}, \\ &f(\tfrac{1}{n}) = \tfrac{1}{n} \\ &f(0) = 0 \end{aligned}$$

Then we have $f \in C(N^*), C_b(N^*)$ and f clearly has $\mathfrak{c}$ square roots.

- Take $f \in C(\mathbb{Q})$ with more than one square root. In particular, there exists some point $x \in \mathbb{Q}$ with $f(x) > 0$. Hence, due to continuity of f, there is some $\varepsilon > 0$ with $\forall y \in B_\varepsilon(x) \cap \mathbb{Q} : f(y) > 0$. But now now we have $|B_\varepsilon(x) \cap (\mathbb{R} \setminus \mathbb{Q})| = \mathfrak{c}$ and for any $y \in B_\varepsilon(x) \cap (\mathbb{R} \setminus \mathbb{Q})$,

$$g_y : \mathbb{Q} \to \mathbb{R}, g_y(z) = \begin{cases} f(z)^{\frac{1}{2}} & z < y \\ -f(z)^{\frac{1}{2}} & z > y \end{cases}$$

is a different square root, so there are $\mathfrak{c}$ many of them. ☐

2. According to 1B.4, $C(\mathbb{R})$ has exactly two idempotents. The idempotents in $C(N^*)$ are precisely the eventually constant sequences consisting only of 0's and 1's, hence there are $\aleph_0$ many of them. We can construct $\mathfrak{c}$ many idempotents in $C(\mathbb{Q})$ by taking $\mathbb{1}_{(x,\infty)}$ for each $x \in \mathbb{R} \setminus \mathbb{Q}$. The idempotents in $C(\mathbb{N})$ are precisely the sequences consisting only of 0's and 1's, hence there are $\mathfrak{c}$ of them. ☐

3. Let $f \in C(\mathbb{Q})$ be nonzero idempotent, i.e. take only the values 0 and 1. Take $x \in \mathbb{Q}$ with $f(x) = 1$, let $\varepsilon > 0$ with $\forall y \in B_\varepsilon(x) \cap \mathbb{Q} : f(y) > 0$ and take any $y \in B_\varepsilon(x) \cap (\mathbb{R} \setminus \mathbb{Q})$. Then $g = f \cdot \mathbb{1}_{(-\infty,y)}$ and $h = f \cdot \mathbb{1}_{(y,\infty)}$ are nonzero idempotents in $C(\mathbb{Q})$ with $f = g + h$.

   In $C(\mathbb{N})$ and $C(N^*)$, only those idempotents with at least two nonzero terms have this property. ☐

4. We give the reasons for non-isomorphism in the following table:

	$C(\mathbb{R})$	$C(\mathbb{Q})$	$C(\mathbb{N})$	$C(N^*)/C_b(N^*)$	$C_b(\mathbb{R})$	$C_b(\mathbb{Q})$
$C_b(\mathbb{N})$	Corollary 1.8	Corollary 1.8	Corollary 1.8	1C.2	1C.2	1C.1
$C_b(\mathbb{Q})$	Corollary 1.8	Corollary 1.8	Corollary 1.8	1C.2	1C.2	
$C_b(\mathbb{R})$	Corollary 1.8	Corollary 1.8	Corollary 1.8	1C.2		
$C(N^*)/C_b(N^*)$	Corollary 1.8	Corollary 1.8	Corollary 1.8			
$C(\mathbb{N})$	1C.2	1C.1				
$C(\mathbb{Q})$	1C.2					

☐

5.
   - Take any irrational number $x \in \mathbb{R} \setminus \mathbb{Q}$. Then we have the homeomorphism $\mathbb{Q} \cong ((-\infty, x) \cap \mathbb{Q}) \sqcup ((x, \infty) \cap \mathbb{Q}) \cong \mathbb{Q} \sqcup \mathbb{Q}$, and this homeomorphism induces an isomorphism $C(\mathbb{Q}) \cong C(\mathbb{Q}) \oplus C(\mathbb{Q})$.
   - There is an obvious homeomorphism $\mathbb{N} \cong \mathbb{N} \sqcup \mathbb{N}$ given by separating the natural numbers into the odd and even numbers, and this homeomorphism induces an isomorphism $C(\mathbb{N}) \cong C(\mathbb{N}) \oplus C(\mathbb{N})$.
   - The decomposition $N^* = \{1\} \sqcup (\{\frac{1}{2}, \frac{1}{3}, \ldots\} \cup \{0\})$ induces an isomorphsm $C(N^*) \cong \mathbb{R} \oplus C(N^*)$. ☐

6. Consider $R = \{f \in C(\mathbb{R}) \mid f \text{ is constant on } [0, 1]\} \subseteq C(\mathbb{R})$. Then the map

$$C(\mathbb{R}) \to R, f \mapsto \left[x \mapsto \begin{cases} f(x) & x \leq 0 \\ f(0) & x \in [0, 1] \\ f(x-1) & x \geq 1 \end{cases} \right]$$

is a ring isomorphism.

Since $\mathbb{R}$ is connected, $C(\mathbb{R})$ has no proper summand. ☐

1D Divisors of functions

Problem

1. If $Z(f)$ is a neighborhood of $Z(g)$, then f is a multiple of g - that is, $f = hg$ for some $h \in C(X)$. Furthermore, if $X \setminus \operatorname{int} Z(f)$ is compact, then h can be chosen to be bounded.
2. Construct an example in which $Z(f) \supseteq Z(g)$, but f is not a multiple of g.
3. If $|f| \leq |g|^r$ for some real $r > 1$, then f is a multiple of g. Hence if $|f| \leq |g|$, then f^r is a multiple of g for every $r > 1$ for which f^r is defined.

Solution

1. Assume that $Z(g) \subseteq \operatorname{int} Z(f)$. Then define

$$h : X \to \mathbb{R}, h(x) = \begin{cases} \frac{f(x)}{g(x)} & x \notin \operatorname{int} Z(f) \\ 0 & x \in Z(f) \end{cases}$$

Then h is clearly continuous, with $f = hg$ and if $X \setminus \operatorname{int} Z(f)$ is compact, h is bounded. □

2. Take $f(x) = x$ and $g(x) = x^2$ on $\mathbb{R}$. □
3. Assume that there is some $r > 1$ with $|f| \leq |g|^r$. Then define

$$h : X \to \mathbb{R}, h(x) = \begin{cases} \frac{f(x)}{g(x)} & x \notin Z(g) \\ 0 & \text{otherwise} \end{cases}$$

In particular, for $x \notin Z(g)$, we have $|h(x)| = \left|\frac{f(x)}{g(x)}\right| \leq \frac{|g(x)|^r}{|g(x)|} = |g(x)|^{r-1}$, so for some $y \in Z(g)$ with $x \to y$, we have $|h(x)| \leq |g(x)|^{r-1} \to |g(y)|^{r-1} = 0$, showing continuity of h. Hence we have $f = hg$.

In particular, if $|f| \leq |g|$, we have $|f^r| \leq |g|^r$, hence f^r is a multiple of g. □

1E Units

Problem

1. Let $f \in C(X)$. There exists a positive unit u of $C(X)$ such that

$$(-\mathbb{1} \vee f) \wedge \mathbb{1} = uf$$

2. The following are equivalent.
   (1) For every $f \in C(X)$, there exists a unit u of $C(X)$ such that $f = u|f|$.
   (2) For every $g \in C_b(X)$, there exists a unit v of $C(X)$ such that $g = v|g|$.
3. Describe the functions f in $C(\mathbb{N})$ for which there exists a unit u of $C(\mathbb{N})$ satisfying $f = u|f|$. Do the same for $C(\mathbb{Q})$ and $C(\mathbb{R})$.
4. Do the same for the equation $f = k|f|$, where k belongs to $C(X)$ but is not necessarily a unit.

Solution

1. Define

$$u : X \to \mathbb{R}, u(x) = \begin{cases} \frac{1}{|f(x)|} & |f(x)| \geq 1 \\ 1 & |f(x)| \leq 1 \end{cases}$$

Then for $|f(x)| \geq 1$ we have $u(x)f(x) = \operatorname{sign}(f(x))$ and $(-1 \vee f(x)) \wedge 1 = \operatorname{sign}(f(x))$ and for $|f(x)| \leq 1$ we have $u(x)f(x) = f(x)$ and $(-1 \vee f(x)) \wedge 1 = f(x)$. This shows that $(-\mathbb{1} \vee f) \wedge \mathbb{1} = uf$. □

2. (1) $\implies$ (2) Take $g \in C_b(X)$. Then there exists a unit $u \in C(X)$ with $g = u|g|$. Now take $v = (-\mathbb{1} \vee u) \wedge \mathbb{1}$. For $g(x) \neq 0$, we have $u(x) = \pm 1$, so $v(x) = u(x)$. For $g(x) = 0$, the identity is true anways. So we have $g = v|g|$.

   (2) $\implies$ (1) Take $f \in C(X)$. Now we know that there exists a positive unit $w \in C(X)$ with $(-\mathbb{1} \vee f) \wedge \mathbb{1} = wf$. In particular, $wf \in C_b(X)$. Hence there exists a unit v of $C_b(X)$ with $wf = v|wf| = vw|f|$, implying $f = v|f|$. □
3. Omitted.
4. Omitted.

1F C-embedding

Problem

1. Every C_b-embedded zero-set is C-embedded.
2. Let $S \subseteq X$. If every zero-set in S is a zero-set in X, then S is C_b-embedded in X.
3. A discrete zero-set is C_b-embedded if and only if all of its subsets are zero-sets.
4. A subset S of $\mathbb{R}$ is C-embedded if and only if it is closed.
5. If a subset S of X is C-embedded in X, then $C(S)$ is a homomorphic image of $C(X)$.

Solution

1. Let $A \subseteq X$ be a C_b-embedded zero-set. Let B be any zero-set with $A \cap B = \emptyset$. Then by Theorem 1.15, A and B are completely separated. Hence, A is completely separated from every zero-set disjoint from it. So by Theorem 1.18, A is C-embedded. □
2. Assume that every zero-set in S is a zero-set in X. Take any two completely separated sets $A, B \subseteq S$. By Theorem 1.15, A and B are contained in disjoint zero-sets C and D: $A \subseteq C$, $B \subseteq D$, $C \cap D = \emptyset$. By the assumption, C and D are also zero-sets in X, so again by Theorem 1.15, A and B are completely seprated in X. So any two completely separated sets in S are also completely separated in X. By Theorem 1.17, this shows that S is C_b-embedded in X. □
3. Assume that $S \subseteq X$ is discrete and all of its subsets are zero-sets in X. Then by 1F.2, S is C_b-embedded in X.

   On the other hand, if $S \subseteq X$ is a discrete C_b-embedded zero-set, taking any subset $R \subseteq S$, the function $\mathbb{1}_{S \setminus R}$ defined on S admits a continuous extension f to all of X. Since zero-sets are closed under intersection, $S \cap Z(f) = R$ is a zero-set. □
4. Assume that $S \subseteq \mathbb{R}$ is C_b-embedded. Assume that $\operatorname{cl} S \setminus S \neq \emptyset$ and take $x \in \operatorname{cl} S \setminus S$. Then there is some sequence in S with $x_n \to x$ and, taking a subsequence such that $|x_n - x|$ is montone, we can define the following continuous function on S:

   $$f : S \to \mathbb{R}, f(y) = \begin{cases} 1 & |x-y| \geq |x-x_1| \\ (-1)^{n+1}\left(1 - 2\frac{|x-x_n|-|x-y|}{|x-x_n|-|x-x_{n+1}|}\right) & |x-x_n| \geq |x-y| \geq |x-x_{n+1}| \end{cases}$$

   f clearly has no continuous extension to all of $\mathbb{R}$.

   On the other hand, again since $\mathbb{R}$ is a metric space, any closed set $S \subseteq \mathbb{R}$ is C-embedded. □
5. Consider the restriction mapping from $C(X)$ to $C(S)$. This is a ring homomorphism and since S is C-embedded, it is surjective. Hence $C(S)$ is a homomorphic image of $C(X)$. □

1G Pseudocompact spaces

Problem

1. Any continuous image of a pseudocompact space is pseudocompact.
2. X is pseudocompact if and only if $f[X]$ is compact for every f in $C_b(X)$.
3. Let X be a Hausdorff space. If, of any two disjoint closed sets, at least one is compact, or even countably compact, then X is countably compact.
4. If, of any two disjoint zero-sets in X, at least one is compact, or even pseudocompact, then X is pseudocompact.

Solution

1. Let $f : X \to Y$ be a surjective mapping where X is pseudocompact. Let $g \in C(Y)$. Then $g \circ f \in C(X)$ must be bounded, but then f must also be bounded, showing that Y is pseudocompact. □
2. If there is some $f \in C_b(X)$ such that $f[X]$ is not compact, then $f[X]$ must have a limit point which it does not contain, say x. Then, postcomposing with $y \mapsto \frac{1}{x-y}$, we obtain an unbounded mapping on X, contradicting pseudocompactness of X.

   On the other hand, if there is some unbounded function f on X, $\arctan \circ f$ is a function in $C_b(X)$ whose image is not compact. □

3. Let $A \subseteq X$ be infinite. Assume that A has no limit point. Then take any partition $A = B \sqcup C$ where both B and C are infinite. Then neither B nor C have a limit point, hence they are both closed. But then at least one of them must be compact, which is impossible. $\square$

4. Assume that X is not pseudocompact. Then there exists a C-embedded copy $\{x_n\}_{x=1}^{n}$ of $\mathbb{N}$. Extend the function $x_n \mapsto \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$ to a function f on all of $\mathbb{R}$. Let $A = Z(f)$ and $B = Z(1-f)$. Then A and B are disjoint zero-sets in X, so at least one of them must be pseudocompact. But now $\{x_n \mid n \text{ odd}\} \subseteq A$ and $\{x_n \mid n \text{ even}\} \subseteq B$ are C-embedded copies of $\mathbb{N}$, a contradiction. $\square$

1H Basically and extremally disconnected spaces

Problem

A space X is said to be extremally disconnected if every open set has an open closure. X is basically disconnected if every cozero-set has an open closure. Hence any extremally disconnected space is basically disconnected.

1. X is extremally disconnected if and only if every pair of disjoint open sets have disjoint closures. What is the analogous condition for basically disconnected spaces?

2. In an extremally disconnected space, any two disjoint open sets are completely separated. In a basically disconnected space, any two disjoint cozero-sets are completely separated. Equivalently, for every $f \in C(X)$, $\operatorname{pos} f$ and $\operatorname{neg} f$ are completely separated.

3. If X is basically disconnected, then for every $f \in C(X)$, there exists a unit $u \in C(X)$ such that $f = u|f|$.

4. Every dense subspace X of an extremally disconnected space T is extremally disconnected, In fact, disjoint open sets in X have disjoint open closures in T.

5. Every open subspace of an extremally disconnected space is extremally disconnected.

6. X is extremally disconnected if and only if every open subspace is C_b-embedded.

Solution

1. Assume that X is extremally disconnected. If $A, B \subseteq X$ are open sets with $A \cap B = \emptyset$, $\operatorname{cl} A$ is clopen with $\operatorname{cl} A \cap B = \emptyset$. Since $\operatorname{cl} A$ is open, we also have $\operatorname{cl} A \cap \operatorname{cl} B = \emptyset$.

   On the other hand, assume that every pair of disjoint open sets have disjoint closures. Let $A \subseteq X$ be an open set. We intend to show that $\operatorname{cl} A$ is open. Consider $B = X \setminus \operatorname{cl} A$. Then A and B are disjoint open sets, hence their closures disjoint. So $\operatorname{cl} B$ cannot be any larger than B, so B must already be closed, showing that $\operatorname{cl} A$ is open.

   The analogous condition for basically disconnected spaces is that given a cozero-set and an open set which are disjoint, their closures are disjoint:

   Assume that X is basically disconnected. If $A \subseteq X$ is a cozero set and $B \subseteq X$ is an open set with $A \cap B = \emptyset$, $\operatorname{cl} A$ is clopen with $\operatorname{cl} A \cap B = \emptyset$. Since $\operatorname{cl} A$ is open, $\operatorname{cl} A \cap \operatorname{cl} B = \emptyset$.

   On the other hand, assume that given a cozero-set and an open set which are disjoint, their closures are disjoint. Let $A \subseteq X$ be a cozero-set. Take $B = X \setminus \operatorname{cl} A$. Then B is open, so their closures are disjoint. Hence $\operatorname{cl} B$ cannot be any larger than B, so B must be closed, so $\operatorname{cl} A$ is open. $\square$

2. Let $A, B \subseteq X$ with $A \cap B = \emptyset$ be two disjoint open sets in an extremally disconnected space. Since A is open, $\mathbb{1}_{X \setminus \operatorname{cl} A}$ is continuous. Since B is open, $\mathbb{1}_{X \setminus \operatorname{cl} B}$ is continuous. Hence $\operatorname{cl} A$ and $\operatorname{cl} B$ are zero-sets, showing that A and B are contained in disjoint zero-sets and hence completely separated.

   Let $A, B \subseteq X$ with $A \cap B = \emptyset$ be two disjoint cozero-sets in a basically disconnected space. Since A is a cozero-set, $\mathbb{1}_{X \setminus \operatorname{cl} A}$ is continuous. Since B is a cozero-set, $\mathbb{1}_{X \setminus \operatorname{cl} B}$ is continuous. Hence $\operatorname{cl} A$ and $\operatorname{cl} B$ are zero-sets, showing that A and B are contained in disjoint zero-sets and hence completely separated. $\square$

3. Let X be basically disconnected and take $f \in C(X)$. Let $A = \operatorname{pos} f$ and $B = \operatorname{neg} f$. Then A and B are disjoint cozero-sets in a basically disconnected space, hence they are contained in disjoint clopen sets $C \supseteq A$ and $D \supseteq B$. Then we simply define

   $$u : X \to \mathbb{R}, u(x) = \begin{cases} 1 & x \in C \\ -1 & x \in X \setminus C \end{cases}$$

   And now we have $f = u|f|$. $\square$

4. Let T be an extremally disconnected space and $X \subseteq T$ a dense subspace. Let $A, B \subseteq X$ with $A \cap B = \emptyset$ be disjoint open sets in X. This means that we can write $A = C \cap X$ and $B = D \cap X$ with open sets C and D in T. If we had $C \cap D \neq \emptyset$, then due to density of X we would have $C \cap D \cap X \neq \emptyset$. But $C \cap D \cap X = A \cap B = \emptyset$, so in fact $C \cap D = \emptyset$. Hence we have $\operatorname{cl}_X A \subseteq \operatorname{cl}_T A \subseteq \operatorname{cl}_T C$ and $\operatorname{cl}_X B \subseteq \operatorname{cl}_T B \subseteq \operatorname{cl}_T D$ with $\operatorname{cl}_T C \cap \operatorname{cl}_T D = \emptyset$, so A and B have disjoint closures in X, showing that X is extremally disconnected. □
5. Let T be an extremally disconnected space and $X \subseteq T$ an open subspace. Let $A, B \subseteq X$ with $A \cap B = \emptyset$ be disjoint open sets in X. Then, A and B are also open in T, hence we have $\operatorname{cl}_X A \subseteq \operatorname{cl}_T A$ and $\operatorname{cl}_X B \subseteq \operatorname{cl}_T B$, which are disjoint since T is extremally disconnected. □
6. Assume that X is extremally disconnected and let $A \subseteq X$ be an open subspace. Let $B, C \subseteq X$ be completely separated sets. Then there are cozero-sets $D, E \subseteq X$ with $B \subseteq D$, $C \subseteq E$ and $D \cap E = \emptyset$. Then D and E are also disjoint and open in X and hence by 1H.2 completely separated in X. This shows that any two completely separated sets in A are completely separated in X, showing that A is C_b-embedded in X.

   On the other hand, if any open subspace $A \subseteq X$ is C_b-embedded in X, take two disjoint open sets B and C. Then $B \cup C$ is C_b-embedded in X and B and C are clopen subsets of $B \cup C$. Thus B and C are completely separated in $B \cup C$ and hence also completely separated in X. In particular, they are contained in disjoint zero-sets and hence have disjoint closures. This shows that X is extremally disconnected. □

1I Algebra homomorphisms

Problem

Let t be a ring homomorphism from $C(Y)$ or $C_b(Y)$ into $C(X)$.

1. $t(\mathtt{r}) = \mathtt{r} \cdot t(1)$ for each $\mathtt{r} \in \mathbb{R}$
2. t is an algebra homomorphism, i.e. $t(\mathtt{r}g) = \mathtt{r} \cdot t(g)$ for all $r \in \mathbb{R}$ and $g \in C(Y)$.

Solution

1. For each fixed $x \in X$, the mapping $r \mapsto t(\mathtt{r})(x)$ is a homomorphism from $\mathbb{R}$ into $\mathbb{R}$ and hence is either zero or the identity. If it is zero, we have $t(\mathtt{r})(x) = 0$ and $(\mathtt{r} \cdot t(\mathbb{1}))(x) = r \cdot t(\mathbb{1})(x) = r \cdot 0 = 0$. If it is the identity, we have $t(\mathtt{r})(x) = r$ and $(\mathtt{r} \cdot t(\mathbb{1}))(x) = r \cdot t(\mathbb{1})(x) = r \cdot 1 = r$. Hence at all points we have $(\mathtt{r} \cdot t(\mathbb{1}))(x) = t(\mathtt{r})(x)$, showing that $\mathtt{r} \cdot t(\mathbb{1}) = t(\mathtt{r})$. □
2. For $g \in C(Y)$ and $r \in \mathbb{R}$, we have $t(\mathtt{r} \cdot g) = t(\mathtt{r}) \cdot t(g) = \mathtt{r} \cdot t(\mathbb{1}) \cdot t(g) = \mathtt{r} \cdot t(\mathbb{1} \cdot g) = \mathtt{r} \cdot t(g)$. □

1J Preservation or reduction of norm

Problem

For $f \in C_b(X)$, define $\|f\| = \sup_{x \in X} |f(x)|$.

1. $\|f\| = \inf\{r \in \mathbb{R} \mid |f| \leq \mathtt{r}\}$
2. If t is a nonzero homomorphism of $C_b(Y)$ into $C_b(X)$, then $\|t(\mathtt{r})\| = |r|$.
3. $t(\mathtt{r}) \leq \mathtt{r}$ for $r \geq 0$ in $\mathbb{R}$.
4. $\|t(g)\| \leq \|g\|$ for every $g \in C_b(Y)$.
5. If $t(g) \leq \mathtt{r}$, then $t(g) \leq t(\mathtt{r})$.
6. If t is an isomorphism into $C_b(X)$, then $\|t(g)\| = \|g\|$ for all $g \in C_b(Y)$.

Solution

1. We clearly have $|f| \leq \|f\|$ and if $r < \|f\|$, then there is some $x \in X$ with $|f(x)| > r$, hence $|f| \not\leq \mathtt{r}$. This shows that $\|f\| = \inf\{r \in \mathbb{R} \mid |f| \leq \mathtt{r}\}$. □
2. Let $t : C_b(Y) \to C_b(X)$ be a nonzero homomorphism. Then we have $\|t(\mathtt{r})\| = \|\mathtt{r} \cdot t(\mathbb{1})\| = |r|$. □
3. Assume $r \geq 0$. At each $x \in X$, we have $t(\mathtt{r})(x) = r$ or $t(\mathtt{r})(x) = 0$. This shows that $t(\mathtt{r}) \leq \mathtt{r}$. □
4. Take $g \in C_b(Y)$. Then we have $g \leq \|g\|$ and $-g \leq \|g\|$. Since t preserves the order structure, we have $t(g) \leq t(\|g\|) \leq \|g\|$ and $-t(g) \leq t(\|g\|) \leq \|g\|$. This shows that $\|t(g)\| \leq \|g\|$. □

5. Assume that $t(g) \leq \mathbb{r}$. For each $x \in X$, the mapping $r \mapsto t(\mathbb{r})(x)$ is either the identity or zero. If it is the identity, we have $t(\mathbb{r})(x) = r$, hence $t(g)(x) \leq t(\mathbb{r})(x)$. If it is zero, we have some $p, q \in \mathbb{R}$ with $\mathbb{p} \leq g \leq \mathbb{q}$ and hence $t(\mathbb{p})(x) \leq t(g)(x) \leq t(\mathbb{q})(x)$, so $0 \leq t(g)(x) \leq 0$, implying $t(g)(x) = 0$. This shows that $t(g) \leq t(\mathbb{r})$. □

6. If t is an isomorphism, we have $\|t(g)\| \leq \|g\|$ and $\|g\| = \|t^{-1}(t(g))\| \leq \|t(g)\|$, showing that $\|t(g)\| = \|g\|$. □

2 Ideals and z-filters

2A Bounded functions in ideals

Problem

The functions f, $f \cdot (1+f^2)^{-1}$ and $(-\mathbb{1} \vee f) \wedge \mathbb{1}$ belong to exactly the same ideals in $C(X)$. Hence every ideal in $C(X)$ has a set of bounded generators.

Solution

The functions f and $f \cdot (1+f^2)^{-1}$ differ only by the unit $1+f^2$. Due to 1E.1, the functions f and $(-\mathbb{1} \vee f) \wedge \mathbb{1}$ differ only by a unit. Hence all these elements belong to exactly the same ideals. Given some ideal I, $\{f \cdot (1+f^2)^{-1} \mid f \in I\}$ is a bounded set of generators for I. □

2B Prime ideals

Problem

1. An ideal P in $C(X)$ is prime if and only if $P \cap C_b(X)$ is a prime ideal in $C_b(X)$.
2. If P and Q are prime ideals in $C(X)$, or in $C_b(X)$, then $PQ = P \cap Q$. In particular, $P^2 = P$. Hence $M^2 = M$ for every maximal ideal M in $C(X)$ or $C_b(X)$.
3. An ideal I in a commutative ring is an intersection of prime ideals if and only if $a^2 \in I$ implies $a \in I$.

Solution

1. Assume that P is a prime ideal in $C(X)$. Take $f, g \in C_b(X)$ with $fg \in P \cap C_b(X)$. Then we have either $f \in P$ or $g \in P$. Hence $P \cap C_b(X)$ is a prime ideal in $C_b(X)$.

   On the other hand, assume that $P \cap C_b(X)$ is a prime ideal in $C_b(X)$. Take $f, g \in C(X)$ with $fg \in P$. Then there are units u and v of $C(X)$ with $fu \in C_b(X)$ and $gv \in C_b(X)$. Hence we have $(fu)(gv) = (fg)(uv) \in P$. We also have $(fu)(gv) \in C_b(X)$, hence $(fu)(gv) \in P \cap C_b(X)$. Since $P \cap C_b(X)$ is a prime ideal, we have either $fu \in P \cap C_b(X)$ or $gv \in P \cap C_b(X)$. If $fu \in P$, we also have $f \in P$ and if $gv \in P$, we also have $g \in P$. This shows that P is a prime ideal in $C(X)$. □

2. Let P and Q be prime ideals in $C(X)$. Clearly, we have $PQ \subseteq P \cap Q$. Now take any $f \in P \cap Q$. Then we have $(f^{\frac{1}{3}})^3 = f \in P$ and since P is prime, we have $f^{\frac{1}{3}} \in P$. Analogously, we also have $f^{\frac{1}{3}} \in Q$. This shows that $f = f^{\frac{1}{3}} \cdot (f^{\frac{1}{3}})^2 \in PQ$. Hence $P \cap Q \subseteq PQ$.

   In particular, we have $P^2 = P \cap P = P$. Also, every maximal ideal is prime, so we have $M^2 = M$. □

3. Assume that $I = \bigcap \mathscr{P}$, where $\mathscr{P}$ is a collection of prime ideals. Take a with $a^2 \in I$. Then we have $a^2 \in P$ for each $P \in \mathscr{P}$. Since P is prime, we have $a \in P$, showing that $a \in I$.

   On the other hand, assume that $a^2 \in I$ implies $a \in I$. Let $\mathscr{P}$ be the collection of all prime ideals containing I. We need to show that given any element b not in I, there is some $P \in \mathscr{P}$ not containing b. Taking such a $b \notin I$, by assumption, we also have $b^n \notin I$ for any $n \in \mathbb{N}$. Hence by Theorem 0.16, there exists a prime ideal P containing I but containing no power of b. In particular, P does not contain b. □

2C Functions congruent to constants

Problem

1. Let I be an ideal in $C(X)$; if $f \equiv \mathbb{r} \pmod I$, then $r \in f[X]$.
2. Let I be an ideal in $C_b(X)$; if $f \equiv \mathbb{r} \pmod I$, then $r \in \operatorname{cl}_{\mathbb{R}} f[X]$.

Solution

1. Take an ideal I in $C(X)$ and assume that $f \equiv \mathbb{r} \pmod I$. Then we know that $f - \mathbb{r} \in I$. Since I is a proper ideal, it contains no units, hence $f - \mathbb{r}$ must have a zero at some point $x \in X$. Hence $f(x) = \mathbb{r}(x) = r$, so $r \in f[X]$. □

2. Take an ideal I in $C_b(X)$ and assume that $f \equiv \mathbb{r} \pmod I$. Then we know that $f - \mathbb{r} \in I$. Since I is a proper ideal, it contains no units, hence $f - \mathbb{r}$ must not be bounded below. Hence there must be some sequence $(x_n)_{n \in \mathbb{N}}$ in X with $f(x_n) - r \to 0$, meaning that $f(x_n) \to r$, hence $r \in \operatorname{cl}_{\mathbb{R}} f[X]$. □

2D z-ideals

Problem

1. Let I be a z-ideal in $C(X)$, and suppose that $f \equiv \mathbf{r} \pmod I$. If $g(x) = r$ wherever $f(x) = r$, then $g \equiv \mathbf{r} \pmod I$.
2. If $f^2 + g^2$ belongs to a z-ideal I, then $f \in I$ and $g \in I$.
3. If I and J are z-ideals, than $IJ = I \cap J$.
4. $Z[(I,J)]$ is the set of all $Z_1 \cap Z_2$, where $Z_1 \in Z[I]$ and $Z_2 \in Z[J]$.

Solution

1. We have $Z(g - \mathbf{r}) = Z(f - \mathbf{r})$. Since $f - \mathbf{r} \in I$ and I is a z-ideal, we also have $g - \mathbf{r} \in I$. □
2. We have $Z(f^2 + g^2) = Z(f) \cap Z(g) \subseteq Z(f), Z(g)$. Since $Z[I]$ is a z-filter, we have $Z(f), Z(g) \in Z[I]$. Since I is a z-ideal, we have $f, g \in I$. □
3. Clearly we have $IJ \subseteq I \cap J$. Now take any $f \in I \cap J$. Then we have $f = f^{\frac{1}{3}} f^{\frac{1}{3}} f^{\frac{1}{3}}$. Since I and J are z-ideals, we also have $f^{\frac{1}{3}} \in I \cap J$. Hence $f \in IJ$. This shows that $I \cap J \subseteq IJ$. □
4. It is clear that $\{Z_1 \cap Z_2 \mid Z_1 \in Z[I] \text{ and } Z_2 \in Z[J]\}$ is just the z-filter generated by $Z[I]$ and $Z[J]$, i.e. $\{Z_1 \cap Z_2 \mid Z_1 \in Z[I] \text{ and } Z_2 \in Z[J]\} = \langle Z[I], Z[J] \rangle$. Now it remains to show that $Z[(I,J)] = \langle Z[I], Z[J] \rangle$. But this is rather clear: $Z[(I,J)]$ contains both $Z[I]$ and $Z[J]$, hence $Z[(I,J)] \supseteq \langle Z[I], Z[J] \rangle$, and given any $f \in I$ and $g \in J$, $Z(f+g) \supseteq Z(f) \cap Z(g) \in \langle Z[I], Z[J] \rangle$, thus $Z(f+g) \in \langle Z[I], Z[J] \rangle$ and hence $Z[(I,J)] \subseteq \langle Z[I], Z[J] \rangle$. □

2E Prime z-filters

Problem

The following are equivalent for a z-filter $\mathscr{F}$.

(1) $\mathscr{F}$ is prime.

(2) Whenever the union of two zero-sets is all of X, at least one of them belongs to $\mathscr{F}$.

(3) Given $Z_1, Z_2 \in Z(X)$, there exists $Z \in \mathscr{F}$ such that one of $Z \cap Z_1$, $Z \cap Z_2$ contains the other.

Solution

The proof proceeds analogously to Theorem 2.9.

(1) $\Longrightarrow$ (2) Trivial.

(2) $\Longrightarrow$ (3) Take $Z(f), Z(g) \in Z(X)$. Then we consider the functions $h = |f| - |g|$, $k = h \vee 0$ and $\ell = h \wedge 0$. Clearly, $Z(k) \cup Z(\ell) = X$. So, by assumption, we have either $Z(k) \in \mathscr{F}$ or $Z(\ell) \in \mathscr{F}$. Assume w.l.o.g. that $Z(\ell) \in \mathscr{F}$. Note that for all $x \in Z(\ell)$, $h(x) \geq 0$. In particular, taking any $x \in Z(\ell) \cap Z(f)$, we also have $x \in Z(g)$. This shows that $Z(\ell) \cap Z(f) \subseteq Z(\ell) \cap Z(g)$.

(3) $\Longrightarrow$ (1) Take a zero-set $Z \in \mathscr{F}$ and zero-sets $Z_1, Z_2 \in Z(X)$ with $Z_1 \cup Z_2 = Z$. We intend to show that either $Z_1 \in \mathscr{F}$ or $Z_2 \in \mathscr{F}$. By assumption, there is a zero-set $Z_3 \in \mathscr{F}$ such that, w.l.o.g., $Z_3 \cap Z_1 \supseteq Z_3 \cap Z_2$. Thus,

$$Z_3 \cap Z_1 = (Z_3 \cap Z_1) \cup (Z_3 \cap Z_2) = Z_3 \cap (Z_1 \cup Z_2) = Z_3 \cap Z \in \mathscr{F}$$

Due to $Z_1 \supseteq Z_3 \cap Z_1$, this shows that $Z_1 \in \mathscr{F}$. □

2F Finite spaces

Problem

Let X be a finite discrete spaces. In $C(X)$:

1. f is a multiple of g if and only if $Z(f) \supseteq Z(g)$.
2. Every ideal is a z-ideal.
3. Every ideal is principal, and, in fact, is generated by an idempotent.
4. Every ideal is an intersection of maximal ideals. The intersection of all the maximal ideals is 0.

5. Every prime ideal is maximal.

Solution

1. Apply 1D.1, using the fact that every set is open. $\square$
2. Follows trivially from 2F.1. $\square$
3. A z-filter $\mathscr{F}$ in $C(X)$ must be fixed and in fact must be of the form $\mathscr{F} = \{B \subseteq X \mid A \subseteq B\}$. Hence the corresponding z-ideal is generated by the idempotent $\mathbb{1}_{X \setminus A}$. $\square$
4. Taking any ideal I, we know that there is some subset $A \subseteq X$ with $I = \left(\mathbb{1}_{X\setminus A}\right) = \bigcap_{p \in A} M_p$. Also, clearly, $\bigcap_{p\in A} M_p = 0$. $\square$
5. A prime z-filter on a discrete space is just a z-ultrafilter. Hence every prime z-ideal is a maximal ideal. $\square$

2G Prime vs. z-ideals in $C(\mathbb{R})$

Problem

1. Select a function ℓ in $C(\mathbb{R})$ such that $\ell(0) = 0$, while $\lim_{x\to 0}\left|\frac{\ell^n(x)}{x}\right| = \infty$ for all $n \in \mathbb{N}$. Apply 0.17 to construct a prime ideal in $C(\mathbb{R})$ that contains $\mathbb{i}$ but not ℓ. This prime ideal is not a z-ideal (and hence is not maximal).
2. Let O_0 denote the ideal of all functions f in $C(\mathbb{R})$ for which $Z(f)$ is a neighbourhood of 0. Define s in $C(\mathbb{R})$ as follows: $s(x) = x \sin\left(\frac{\pi}{x}\right)$ for $x \neq 0$, and $s(0) = 0$. Then (O_0, s) is not a z-ideal; and the smallest z-ideal containing (O_0, s) is not prime.

Solution

1. We take

$$\ell(x) = \begin{cases} \frac{1}{\ln\left(\frac{1}{|x|}\right)} & -\frac{1}{e} \leq x \leq \frac{1}{e} \\ 1 & x \leq -\frac{1}{e} \text{ or } \frac{1}{e} \leq x \end{cases}$$

   Then,

$$\lim_{x\to 0} f(x) = \lim_{x \to 0} \frac{1}{\ln\left(\frac{1}{|x|}\right)} = \lim_{t\to\infty} \frac{1}{\ln(|t|)} = 0$$

   while

$$\lim_{x\to 0}\left|\frac{\ell^n(x)}{x}\right| = \lim_{t\to\infty}\left|t \cdot \ell^n\left(\frac{1}{t}\right)\right| = \lim_{t\to\infty}\left|t\cdot\frac{1}{\ln(|t|)^n}\right| = \infty$$

   Now consider the principal ideal $I = (\mathbb{i})$. As the above calculation shows, no power of ℓ belongs to I. Hence, by 0.17, there exists a prime ideal P containing I but not ℓ. Due to $Z(\mathbb{i}), Z(\ell) = \{0\}$, P is not a z-ideal. $\square$

2. Consider the function $f(x) = \sqrt{|x|}\sin\left(\frac{\pi}{x}\right)$. It is clear that $Z(f) = Z(s)$. However, if we were able to write $f = g + h \cdot s$ for $g \in O_0$ and $h \in C(\mathbb{R})$, that would imply that $f = h \cdot s$ on some neighbourhood of 0, meaning that f would be a multiple of s on some neighbourhood of 0, which is clearly wrong. Hence (O_0, s) is not a z-ideal. Next, let I be the smallest z-ideal containing (O_0, s). More precisely, due to 2D.1, I is the z-ideal associated to the z-filter consisting precisely of those sets containing a subset of $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \cup \{0\} \cup \{\ldots, -\frac{1}{3}, -\frac{1}{2}, 1\}$ that contains eventually all elements. This ideal is not prime since the functions $\mathbb{i} \vee 0$ and $\mathbb{i} \wedge 0$ do not belong to I, but their product does. $\square$

2H The identity function $\mathbb{i}$ in $C(\mathbb{R})$

Problem

1. The principal ideal $(\mathbb{i})$ in $C(\mathbb{R})$ consists precisely of all functions in $C(\mathbb{R})$ that vanish at 0 and have a derivative at 0. Hence every nonnegative function in $(\mathbb{i})$ has a zero derivative at 0.
2. $(\mathbb{i})$ is not a prime ideal; in fact, $(\mathbb{i})^2 \neq (\mathbb{i})$.
3. The ideal $(\mathbb{i}, |\mathbb{i}|)$ is not principal.
4. Exhibit a principal ideal containing $(\mathbb{i}, |\mathbb{i}|)$.

Solution

1. If $f(x) = x \cdot g(x)$, then we have $f(0) = 0$ and $f'(0) = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} g(h) = g(0)$. Hence f vanishes at 0 and is differentiable at 0.

   On the other hand, assume that f vanishes at 0 and is differentiable at 0. Define

   $$g(x) = \begin{cases} \frac{f(x)}{x} & x \neq 0 \\ f'(0) & x = 0 \end{cases}$$

   By definition of differentiability, $g \in C(\mathbb{R})$, and clearly, $f(x) = x \cdot g(x)$, showing that $f \in (\mathbb{i})$.

   Now, a nonnegative function in $(\mathbb{i})$ must have a local minimum at 0 and hence have a zero derivative at 0. □

2. Take $f, g \in C(\mathbb{R})$ vanishing at 0 and possessing a zero derivative at 0. Then $(fg)'(0) = f'(0)g(0) + f(0)g'(0) = 0$. Thus all elements of $(\mathbb{i})^2$ have a zero derivative at 0, showing that $(\mathbb{i})^2 \subsetneq (\mathbb{i})$. Due to 2B.2, this shows that $(\mathbb{i})$ is not a prime ideal. □

3. Striving for a contradiction, assume that $(\mathbb{i}, |\mathbb{i}|) = (d)$ for some $d \in C(\mathbb{R})$. In particular, there exist $g, h \in C(\mathbb{R})$ with $\mathbb{i} = dg$ and $|\mathbb{i}| = dh$. This implies that $dg = dh$ on $[0, \infty)$ and $dg = -dh$ on $(-\infty, 0]$. In particular, we have $g = h$ on $(0, \infty)$ and $g = -h$ on $(-\infty, 0)$. Due to continuity, this implies that $g(0), h(0) = 0$.

   Next, we must have $s, t \in C(\mathbb{R})$ with $s\mathbb{i} + t|\mathbb{i}| = d$. Hence we have $sgd + thd = s\mathbb{i} + t|\mathbb{i}| = d$. Due to continuity, this implies $sg + th = \mathbb{1}$. But both g and h vanish at 0, a contradiction. Hence $(\mathbb{i}, |\mathbb{i}|)$ cannot be principal. □

4. Take $f(x) = \sqrt{|x|}$ and $g(x) = \operatorname{sign}(x)\sqrt{|x|}$. Then $f^2 = |\mathbb{i}|$ and $fg = \mathbb{i}$, showing that $\mathbb{i}, |\mathbb{i}| \in (f)$ and hence $(\mathbb{i}, |\mathbb{i}|) \subseteq (f)$. □

2I $C(\mathbb{Q})$ and $C_b(\mathbb{Q})$

Problem

The set of all f in $C(\mathbb{Q})$ for which $\lim_{x\to\pi} f(x) = 0$ is not an ideal in $C(\mathbb{Q})$. But the bounded functions in this set do constitute an ideal in $C_b(\mathbb{Q})$.

Solution

Let $I = \{f \in C(\mathbb{Q}) \mid \lim_{x\to\pi} f(x) = 0\}$. Then with $f(x) = \frac{1}{x-\pi}$ and $g(x) = x - \pi$, we have $g \in I$ and $f \in C(\mathbb{Q})$, but $fg \notin I$. Hence I is not an ideal in $C(\mathbb{Q})$.

Now, set $J = \{f \in C_b(\mathbb{Q}) \mid \lim_{x\to\pi} f(x) = 0\}$. If $f \in J$ and $g \in C_b(\mathbb{Q})$ with $|g(x)| \leq C$, then $\lim_{x\to\pi} |f(x)g(x)| \leq \lim_{x\to\pi} |f(x)| \cdot C = 0$, showing that $fg \in J$. Hence J is an ideal in $C_b(\mathbb{Q})$. □

2J Ideal chains in $C(\mathbb{R})$, $C(\mathbb{Q})$ and $C(\mathbb{N})$

Problem

1. Find a chain of z-ideals in $C(\mathbb{R})$ (under set inclusion) that is in one-one, order-preserving correspondence with $\mathbb{R}$ itself.
2. Find a chain of z-ideals in $C(\mathbb{Q})$ in one-one, order-preserving correspondence with $\mathbb{R}$.
3. Do the same for $C(\mathbb{N})$.

Solution

1. Map the real number r to the z-ideal I_r corresponding to the z-filter $\mathscr{F}_r$ of all sets containing $[r, \infty)$. Then we have

   $$r \leq s \Leftrightarrow [r, \infty) \supseteq [s, \infty) \Leftrightarrow \mathscr{F}_r \subseteq \mathscr{F}_s \Leftrightarrow I_r \subseteq I_s$$

   This shows that the correspondence is order-preserving and it is clearly one-one. □

2. Map the real number r to the z-ideal I_r corresponding to the z-filter $\mathscr{F}_r$ of all sets containing $[r, \infty)$. Then we have

   $$r \leq s \Leftrightarrow [r, \infty) \supseteq [s, \infty) \Leftrightarrow \mathscr{F}_r \subseteq \mathscr{F}_s \Leftrightarrow I_r \subseteq I_s$$

   This shows that the correspondence is order-preserving and it is clearly one-one. □

3. Take any bijection $\varphi : \mathbb{N} \to \mathbb{Q}$. From (2.), we know that there exists a chain $\{Q_r\}_{r\in\mathbb{R}}$ of subsets of $\mathbb{Q}$ that is order-anti-isomorphic to $\mathbb{R}$. Now take $N_r = \varphi^{-1}(Q_r)$, $\mathscr{F}_r = \{A \subseteq \mathbb{N} \mid N_r \subseteq A\}$ and $I_r = Z^{\leftarrow}[\mathscr{F}_r]$. Then we have
$$r \le s \Leftrightarrow Q_r \supseteq Q_s \Leftrightarrow N_r \supseteq N_s \Leftrightarrow \mathscr{F}_r \subseteq \mathscr{F}_s \Leftrightarrow I_r \subseteq I_s$$
This shows that the correspondence is order-preserving and it is clearly one-one. □

2K z-filters and C_b

Problem

If M is a maximal ideal in $C_b(X)$, and $Z[M]$ is a z-filter, then $Z[M]$ is a z-ultrafilter.

Solution

Take some z-filter $\mathscr{F}$ with $Z[M] \subseteq \mathscr{F}$ and let $I = Z^{\leftarrow}[\mathscr{F}]$. Since $\mathscr{F}$ is a z-filter, I is a z-ideal in $C_b(X)$. Additionally, we have $I \supseteq M$. Since M was maximal, this implies that $I = M$ and hence that $\mathscr{F} = Z[M]$. Hence $Z[M]$ is a z-ultrafilter. □

2L e-filters and e-ideals

Problem

This problem contains an outline for a theory of z-filters applicable to $C_b(X)$. For $f \in C_b(X)$ and $\varepsilon > 0$, we define
$$E_\varepsilon(f) = f^{\leftarrow}[[-\varepsilon, \varepsilon]] = \{x \mid |f(x)| \le \varepsilon\}$$
Every such set is a zero-set; conversely, every zero-set is of this form: $Z(g) = E_\varepsilon(\boldsymbol{\varepsilon} + |g|)$. For $I \subseteq C_b(X)$, we write
$$E(I) = \{E_\varepsilon(f) \mid f \in I, \varepsilon > 0\}$$
i.e. $E(I) = \bigcup_{\varepsilon>0} E_\varepsilon[I]$. Finally, for any family $\mathscr{F}$ of zero-sets, we define
$$E^-(\mathcal{F}) = \{f \in C_b(X) \mid E_\varepsilon(f) \in \mathscr{F} \text{ for all } \varepsilon > 0\}$$
that is, $E^-(\mathcal{F}) = \bigcap_{\varepsilon>0} E_\varepsilon{}^{\leftarrow}[\mathscr{F}]$.

1. $\mathscr{F} \supseteq E(E^-(\mathscr{F})) = \bigcup_{\varepsilon>0}\{E_\varepsilon(f) \mid E_\delta(f) \in \mathscr{F} \text{ for all } \delta > 0\}$. Note that the inclusion may be proper, even when $\mathscr{F}$ is a z-filter.
2. A z-filter $\mathscr{F}$ is called an e-filter if $E(E^-(\mathscr{F})) = \mathscr{F}$. Hence $\mathscr{F}$ is an e-filter if and only if, whenever $Z \in \mathscr{F}$, there exist f and ε such that $E_\delta(f) \in \mathscr{F}$ for every $\delta > 0$, and $Z = E_\varepsilon(f)$.
3. $I \subseteq E^-(E(I)) = \{f \mid E_\varepsilon(f) \in E(I) \text{ for all } \varepsilon > 0\}$. Note that the inclusion may be proper, even when I is an ideal.
4. An ideal I in $C_b(X)$ is called an e-ideal if $E^-(E(I)) = I$. Hence I is an e-ideal if and only if, whenever $E_\varepsilon(f) \in E(I)$ for all $\varepsilon > 0$, then $f \in I$. Intersections of e-ideals are e-ideals.
5. If I is an ideal in $C_b(X)$, then $E(I)$ is an e-filter. The corresponding result holds in $C(X)$.
6. If $\mathscr{F}$ is any z-filter, then $E^-(\mathscr{F})$ is an ideal in $C_b(X)$. Note, however, that the corresponding result may fail in $C(X)$, even if $\mathscr{F}$ is an e-filter.
7. $I \subseteq J$ implies $E(I) \subseteq E(J)$, and $\mathscr{F} \subseteq \mathscr{G}$ implies $E^-(\mathscr{F}) \subseteq E^-(\mathscr{G})$.
8. If J is an e-ideal, then $I \subseteq J$ if and only if $E(I) \subseteq E(J)$. If $\mathscr{F}$ is an e-filter, then $\mathscr{F} \subseteq \mathscr{G}$ if and only if $E^-(\mathscr{F}) \subseteq E^-(\mathscr{G})$.
9. If $\mathscr{F}$ is any e-filter, then $E^-(\mathscr{F})$ is an e-ideal. If I is any ideal in $C_b(X)$, then $E^-(E(I))$ is the smallest e-ideal containing I. In particular, every maximal ideal in $C_b(X)$ is an e-ideal.
10. For any z-filter $\mathscr{G}$, $E(E^-(\mathscr{G}))$ is the largest e-filter contained in $\mathscr{G}$.
11. If $\mathscr{A}$ is a z-ultrafilter, and a zero-set Z meets every member of $E(E^-(\mathscr{A}))$, then $Z \in \mathscr{A}$.
12. Every e-filter is contained in an e-ultrafilter.
13. If M_b is a maximal ideal in $C_b(X)$, then $E(M_b)$ is an e-ultrafilter; and if $\mathscr{E}$ is an e-ultrafilter, then $E^-(\mathscr{E})$ is a maximal ideal in $C_b(X)$. Hence the correspondence $M_b \mapsto E(M_b)$ is one-one from the set of all maximal ideals in $C_b(X)$ onto the set of all e-ultrafilters.

14. The following property characterizes an ideal M_b in $C_b(X)$ as a maximal ideal: Given $f \in C_b(X)$, if every $E_\varepsilon(f)$ meets every member of $E(M_b)$, then $f \in M_b$.

15. If $\mathscr{A}$ is a z-ultrafilter, then it is the unique z-ultrafilter containing $E(E^-(\mathscr{A}))$. Moreover, $E(E^-(\mathscr{A}))$ is an e-ultrafilter, and it is the unique one contained in $\mathscr{A}$. Hence the correspondence $\mathscr{A} \mapsto E(E^-(\mathscr{A}))$ is one-one from the set of all z-ultrafilters onto the set of all e-ultrafilters.

16. If $\mathscr{A}$ is a z-ultrafilter, then $E^-(\mathscr{A})$ is the maximal ideal
$$E^-(E(E^-(\mathscr{A})))$$
in $C_b(X)$. Hence the correspondence
$$M \mapsto E^-(Z[M])$$
is one-one from the set of all maximal ideals in $C(X)$ onto the set of all maximal ideals in $C_b(X)$. Its inverse is the correspondense $M_b \mapsto Z^{\leftarrow}[\mathscr{A}]$, where $\mathscr{A}$ is the unique z-ultrafilter containing the e-ultrafilter $E(M_b)$.

Solution

1. First of all, we will show that
$$E(E^-(\mathscr{F})) = \bigcup_{\varepsilon>0} \{E_\varepsilon(f) \mid E_\delta(f) \in \mathscr{F} \text{ for all } \delta > 0\}$$
Indeed, we clearly have
$$\begin{aligned} E(E^-(\mathscr{F})) &= \textstyle\bigcup_{\varepsilon>0} E_\varepsilon[E^-(\mathscr{F})] \\ &= \textstyle\bigcup_{\varepsilon>0}\{E_\varepsilon(f) \mid f \in E^-(\mathscr{F})\} \\ &= \textstyle\bigcup_{\varepsilon>0}\{E_\varepsilon(f) \mid E_\delta(f) \in \mathscr{F} \text{ for all } \delta > 0\} \end{aligned}$$
Next, we will show that
$$\mathscr{F} \supseteq E(E^-(\mathscr{F}))$$
This is now clear, since given any $Z \in E(E^-(\mathscr{F}))$, it must be of the form $Z = E_\varepsilon(f)$ for some $f \in E^-(\mathscr{F})$. Since $f \in E^-(\mathscr{F})$, by definition, $Z = E_\varepsilon(f) \in \mathscr{F}$, showing that $E(E^-(\mathscr{F})) \subseteq \mathscr{F}$.

   Next, let's give an example of when this inclusion is proper, even when $\mathscr{F}$ is a z-filter: Let $\mathscr{F}$ be the principal z-filter on $\mathbb{R}$ generated by 0. Then $E^-(\mathscr{F})$ is the set of all $f \in C_b(\mathbb{R})$ such that $f(0) \in [-\varepsilon, \varepsilon]$ for all $\varepsilon > 0$, i.e. $f(0) = 0$. But now $E(E^-(\mathscr{F}))$ consists of sets of the form $f^{\leftarrow}[[-\varepsilon, \varepsilon]]$, which, due to continuity of f, are neighbourhoods of 0. Hence in this case, $\mathscr{F} \supsetneq E(E^-(\mathscr{F}))$ □

2. Follows trivially from (1). □

3. First of all,
$$E^-(E(I)) = \{f \mid E_\varepsilon(f) \in E(I) \text{ for all } \varepsilon > 0\}$$
is true literally by definition of $E^-(E(I))$.

   Next, let's try to see why
$$I \subseteq E^-(E(I))$$
Taking any $f \in I$ and any $\varepsilon > 0$, we wish to see that $E_\varepsilon(f) \in E(I)$. But this is true by definition of $E(I)$.

   Finally, let's give an example of when the inclusion is proper, even though I is an ideal. To do this, consider $O_0 \subseteq C_b(\mathbb{R})$, the ideal of all functions vanishing on a neighbourhood of 0. Then $E(I)$ consists of all neighbourhoods of 0, meaning that with $f(x) = (-1 \vee x) \wedge 1$, $f \in E^-(E(O_0))$, even though $f \notin O_0$. This shows that $O_0 \subsetneq E^-(E(O_0))$. □

4. The only thing that does not follow trivially from (3) is the fact that intersections of e-ideals are e-ideals. Let $\mathscr{E}$ be a collection of e-ideals. Let $J = \bigcap \mathscr{E}$. Take any $f \in C_b(X)$ such that for all $\varepsilon > 0$, we have $E_\varepsilon(f) \in E(J)$. Then for $I \in \mathscr{E}$ we have $E_\varepsilon(f) \in E(J) \subseteq E(I)$. This implies that $f \in I$ and since I was arbitrary, we have $f \in J$, implying that any intersection of e-ideals is an e-ideal. □

5. Let I be an ideal in $C_b(X)$. We intend to show that $E(I)$ is an e-filter. We know that we generally have $E(I) \supseteq E(E^-(E(I)))$. We also know that we generally have $I \subseteq E^-(E(I))$, implying $E(I) \subseteq E(E^-(E(I)))$. In total, we have $E(I) = E(E^-(E(I)))$. Thus, to show that $E(I)$ is an e-filter, it only remains to show that $E(I)$ is even a z-filter.

   That $\emptyset \notin E(I)$ is clear, since otherwise we would have some $f \in I$ which is bounded below and hence invertible, contradicting the fact that I is a proper ideal.

Next, take $E_\varepsilon(f), E_\delta(g) \in E(I)$. Assume w.l.o.g. that $f, g \geq 0$. We need to show that there is some $E_\gamma(h) \subseteq E_\varepsilon(f) \cap E_\delta(g)$. For this, simply take $h = f + g$ and $\gamma = \min(\varepsilon, \delta)$. We see that $h \in I$. Then for any $x \in X$ with $h(x) \leq \gamma$ we have $f(x) \leq h(x) \leq \gamma \leq \varepsilon$ and analogously $g(x) \leq h(x) \leq \gamma \leq \delta$, showing that $E_\gamma(h) \subseteq E_\varepsilon(f) \cap E_\delta(g)$ with $E_\gamma(h) \in E(I)$.

Finally, take $E_\varepsilon(f) \in E(I)$ and any $Z(g) \in E(I)$ with $E_\varepsilon(f) \subseteq Z(g)$. Assume w.l.o.g. that $f, g \geq 0$. Define

$$h : X \to \mathbb{R}, h(x) = \begin{cases} 1 & x \in E_\varepsilon(f) \\ g(x) + \frac{\varepsilon}{f(x)} & x \notin E_\varepsilon(f) \end{cases}$$

Clearly, h is continuous and bounded. Additionaly, we have

$$(fh)(x) = \begin{cases} f(x) & x \in E_\varepsilon(f) \\ f(x)g(x) + \varepsilon & x \notin E_\varepsilon(f) \end{cases}$$

Then $E_\varepsilon(fh) = E_\varepsilon(f) \cup Z(g) = Z(g)$. Due to $fh \in I$, this shows that $Z(g) \in E(I)$.

The analogous proof works for $C(X)$. □

6. Let $\mathscr{F}$ be a z-filter.

   First of all, $E^-(\mathscr{F})$ contains no unit of $C_b(X)$ - otherwise, $\mathscr{F}$ would have to contain $\emptyset$.

   Next, take $f, g \in E^-(\mathscr{F})$. Take $\varepsilon > 0$. We need to show that $E_\varepsilon(f + g) \in \mathscr{F}$. But we have $E_\varepsilon(f + g) \supseteq E_{\frac{\varepsilon}{2}}(f) \cap E_{\frac{\varepsilon}{2}}(g) \in \mathscr{F}$. Since $\mathscr{F}$ is a z-filter, this also shows that $E_\varepsilon(f + g) \in \mathscr{F}$ and hence $f + g \in E^-(\mathscr{F})$.

   Finally, if $f \in E^-(\mathscr{F})$ and $g \in C_b(X)$ with $|g(x)| \leq M$, take $\varepsilon > 0$. Then we have $E_\varepsilon(fg) \supseteq E_{\frac{\varepsilon}{M}}(f) \in \mathscr{F}$, showing that $E_\varepsilon(fg) \in \mathscr{F}$ and hence $fg \in E^-(\mathscr{F})$.

   To see that this fails if we consider $C(X)$, take $\mathscr{F} = \{A \subseteq \mathbb{N} \mid \mathbb{N} \setminus A \text{ is finite}\}$. This is an e-filter since $E^-(\mathscr{F})$ consists precisely of those functions vanishing at infinity and hence we see that $E(E^-(\mathscr{F})) = \mathscr{F}$. However, the functions vanishing at infinity do not constitute an ideal in $C(\mathbb{N})$. □

7. Clear. □

8. That $I \subseteq J$ implies $E(I) \subseteq E(J)$ is clear and if $E(I) \subseteq E(J)$, then $E = E^-(E(I)) \subseteq E^-(E(J)) = J$. Analogously, that $\mathscr{F} \subseteq \mathscr{G}$ implies $E^-(\mathscr{F}) \subseteq E^-(\mathscr{G})$ is clear and if $E^-(\mathscr{F}) \subseteq E^-(\mathscr{G})$, then $\mathscr{F} = E(E^-(\mathscr{F})) \subseteq E(E^-(\mathscr{G})) = \mathscr{G}$. □

9. Let $\mathscr{F}$ be an e-filter. By (6), $E^-(\mathscr{F})$ is an ideal in $C_b(X)$. We also have $\mathscr{F} \supseteq E(E^-(\mathscr{F}))$ and hence $E^-(\mathscr{F}) \supseteq E^-(E(E^-(\mathscr{F})))$ but also $E^-(\mathscr{F}) \subseteq E^-(E(E^-(\mathscr{F})))$, showing that $E^-(\mathscr{F}) = E^-E(E^-(\mathscr{F})))$ and hence $E^-(\mathscr{F})$ is even an e-ideal.

   Let I be any ideal in $C_b(X)$. Then, $E(I)$ is an e-filter and $E^-(E(I))$ is an e-ideal with $I \subseteq E^-(E(I))$. Now, let J be any e-ideal with $I \subseteq J$. Then we have $E^-(E(I)) \subseteq E^-(E(J)) = J$. This shows that $E^-(E(I))$ is the smallest e-ideal containing I.

   If M is a maximal ideal in $C_b(X)$, then $E^-(E(M))$ is an e-ideal containing M, hence we have $M = E^-(E(M))$, showing that M is an e-ideal. □

10. Let $\mathscr{G}$ be a z-filter. Then $E(E^-(\mathscr{G}))$ is an e-filter with $\mathscr{G} \supseteq E(E^-(\mathscr{G}))$. Now if $\mathscr{F}$ is an e-filter with $\mathscr{F} \subseteq \mathscr{G}$, then $\mathscr{F} = E(E^-(\mathscr{F})) \subseteq E(E^-(\mathscr{G}))$, showing that $E(E^-(\mathscr{G}))$ is the largest e-filter containeed in $\mathscr{G}$. □

11. Let $\mathscr{A}$ be a z-ultrafilter and assume that the zero-set Z meets every member of $E(E^-(\mathscr{A}))$. First off, we will show by contradiction that Z meets every member of $\mathscr{A}$. Striving for a contradiction, assume there is some $Z_1 \in \mathscr{A}$ with $Z \cap Z_1 = \emptyset$. Then Z and Z_1 are disjoint zero-sets and hence completely separated by a function f with $f|_Z = 1$ and $f|_{Z_1} = 0$. We intend to show that $f \in E^-(\mathscr{A})$. So, take $\varepsilon > 0$. Then we have $E_\varepsilon(f) \supseteq Z(f) \supseteq Z_1 \in \mathscr{A}$. This shows that $E_\varepsilon(f) \in \mathscr{A}$ and hence $f \in E^-(\mathscr{A})$. But now, $E_{\frac{1}{2}}(f) \cap Z = \emptyset$, a contradiction. This shows that Z meets every member of $\mathscr{A}$. But now we must have $Z \in \mathscr{A}$. □

12. We must show that the union of a chain of e-filters is again an e-filter. So, let $\mathscr{E}$ be such a chain. Set $\mathscr{G} = \bigcup \mathscr{E}$. It is already known that $\mathscr{G}$ is a z-filter. Now, take $Z \in \mathscr{G}$. Hence there is some $\mathscr{F} \in \mathscr{E}$ with $Z \in \mathscr{F}$. Since $\mathscr{F}$ is an e-filter, we must be able to write $Z = E_\varepsilon(f)$ such that for every $\delta > 0$, $E_\delta(f) \in \mathscr{F}$. In particular, taking any $\delta > 0$, we have $E_\delta(f) \in \mathscr{G}$. This shows that $\mathscr{G}$ is an e-filter. □

13. Let M_b be a maximal ideal in $C_b(X)$. Let $\mathscr{F}$ be an e-filter with $E(M_b) \subseteq \mathscr{F}$. Due to (9), M_b is actually an e-ideal, showing that $M_b = E^-(E(M_b)) \subseteq E^-(\mathscr{F})$. Since M_b is a maximal ideal, we must have $M_b = E^-(\mathscr{F})$, showing that $E(M_b) = \mathscr{F}$. Hence $E(M_b)$ is an e-ultrafilter.

Let $\mathscr{E}$ be an e-ultrafilter. Let I be an ideal with $E^{-}(\mathscr{E}) \subseteq I$. Then we have $\mathscr{E} = E(E^{-}(\mathscr{E})) \subseteq E(I)$. Due to (5), $E(I)$ is an e-filter and since $\mathscr{E}$ is an e-ultrafilter, it follows that $\mathscr{E} = E(I)$ and hence $E^{-}(\mathscr{E}) = E^{-}(E(I)) \supseteq I$. This shows that $E^{-}(\mathscr{E}) = I$ and hence $E^{-}(\mathscr{E})$ is a maximal ideal in $C_b(X)$.

Now, let's consider the mapping $M_b \mapsto E(M_b)$ from the set of all maximal ideals in $C_b(X)$ to the set of all e-ultrafilters. We have just shown that this mapping is well-defined in the sense that it actually maps maximal ideals of $C_b(X)$ to e-ultrafilters. Next, consider the mapping $\mathscr{E} \mapsto E^{-}(\mathscr{E})$ from the set of all e-ultrafilters to the set of all maximal ideals in $C_b(X)$. We have just shown that this mapping is well-defined in the sense that it actually maps e-ultrafilters to maximal ideals of $C_b(X)$. Now, we should argue why these maps are inverses of each other. First of all, take any e-ultrafilter $\mathscr{E}$. Then, by definition, $E(E^{-}(\mathscr{E})) = \mathscr{E}$. On the other hand, take any maximal ideal M_b of $C_b(X)$. Then by (9), M_b is an e-ideal and hence $E^{-}(E(M_b)) = M_b$. This shows that the two mappings are inverses of each other and hence are both bijective. □

14. $\Longrightarrow$ Let M_b be a maximal ideal in $C_b(X)$ and take $f \in C_b(X)$ such that every $E_\varepsilon(f)$ meets every member of $E(M_b)$. We will show by contradiction that $f \in M_b$. Assume, striving for a contradiction, that $f \notin M_b$. Then consider $I = (M_b, f)$. Take any $g + hf \in (M_b, f)$ with $g \in M_b$ and $h \in C_b(X)$, bounded by M. Then we have, for any $\varepsilon > 0$, $E_\varepsilon(g + hf) \supseteq E_{\frac{\varepsilon}{2}}(g) \cap E_{\frac{\varepsilon}{2}}(hf) \supseteq E_{\frac{\varepsilon}{2}}(g) \cap E_{\frac{\varepsilon}{2M}}(f) \neq \emptyset$. Hence $g + hf$ is not invertible, showing that I is a proper ideal. Due to $M_b \subseteq I$ and the fact that M_b is maximal, we must have $M_b = I$ and hence $f \in M_b$.

    $\Longleftarrow$ On the other hand, let M_b be an ideal in $C_b(X)$ such that, given any $f \in C_b(X)$ with the property that every $E_\varepsilon(f)$ meets every member of $E(M_b)$, we have $f \in M_b$. We intend to show that M_b is maximal. Striving for a contradiction, assume that there is some $f \in C_b(X)$ such that (M_b, f) is a proper ideal with $M_b \subsetneq (M_b, I)$. Since $f \notin M_b$, by assumption, there must be some case in which $E_\varepsilon(f) \cap E_\delta(g) = \emptyset$, where $\varepsilon, \delta > 0$ and $g \in M_b$. W.l.o.g., assume that $f \geq 0$ and $g \geq 0$. Set $\gamma = \min(\varepsilon, \delta)$. But then $E_\gamma(f + g) \subseteq E_\gamma(f) \subseteq E_\varepsilon(f)$ and $E_\gamma(f + g) \subseteq E_\gamma(g) \subseteq E_\delta(g)$. Hence $E_\gamma(f + g) \subseteq E_\varepsilon(f) \cap E_\delta(g) = \emptyset$. Hence $f + g$ is invertible, a contradiction to the fact that (M_b, I) is a proper ideal. This contradiction shows that M_b is maximal. □

15. Let $\mathscr{A}$ and $\mathscr{B}$ be z-ultrafilters such that $\mathscr{B} \supseteq E(E^{-}(\mathscr{A}))$. We intend to show that $\mathscr{A} = \mathscr{B}$. Take any $Z_1 \in \mathscr{A}$ and $Z_2 \in E(E^{-}(\mathscr{A}))$. Then we also have $Z_1 \in \mathscr{B}$. Hence $Z_1 \cap Z_2 \in \mathscr{B}$ and so $Z_1 \cap Z_2 \neq \emptyset$. By (11), this implies that $\mathscr{B} \subseteq \mathscr{A}$ and since $\mathscr{B}$ is a z-ultrafilter we even have $\mathscr{B} = \mathscr{A}$. Hence $\mathscr{A}$ is the unique z-ultrafilter containing $E(E^{-}(\mathscr{A}))$.

    Since $\mathscr{A}$ is a z-filter, by (10), $E(E^{-}(\mathscr{A}))$ is the largest e-filter contained in $\mathscr{A}$. Now, take any e-filter $\mathscr{E}$ with $E(E^{-}(\mathscr{A})) \subseteq \mathscr{E}$. Then $\mathscr{E}$ must be contained in some z-ultrafilter $\mathscr{B}$. But now $\mathscr{B}$ is a z-ultrafilter containing $E(E^{-}(\mathscr{A}))$, showing that $\mathscr{B} = \mathscr{A}$ and hence we have $\mathscr{E} \subseteq \mathscr{A}$. But now, $\mathscr{E}$ is an e-filter contained in $\mathscr{A}$ and since $E(E^{-}(\mathscr{A}))$ is the largest e-filter contained in $\mathscr{A}$, we must in fact have $E(E^{-}(\mathscr{A})) = \mathscr{E}$. This shows that $E(E^{-}(\mathscr{A}))$ is an e-ultrafilter.

    Take any e-ultrafilter $\mathscr{E}$ contained in $\mathscr{A}$. Then we have $\mathscr{E} = E(E^{-}(\mathscr{E})) \subseteq E(E^{-}(\mathscr{A}))$. Since $\mathscr{E}$ is maximal, this shows that $\mathscr{E} = E(E^{-}(\mathscr{A}))$ and hence $E(E^{-}(\mathscr{A}))$ is the unique e-ultrafilter contained in $\mathscr{A}$.

    Consider now the mapping $\mathscr{A} \mapsto E(E^{-}(\mathscr{A}))$ from the set of all z-ultrafilters to the set of all e-ultrafilters. As we have just shown, this mapping is well-defined in the sense that it takes z-ultrafilters to e-ultrafilters and as we have just shown, this mapping is bijective. □

16. Let $\mathscr{A}$ be a z-ultrafilter. Then we have $\mathscr{A} \supseteq E(E^{-}(\mathscr{A}))$ and hence $E^{-}(\mathscr{A}) \supseteq E^{-}(E(E^{-}(\mathscr{A})))$. We also have $E^{-}(\mathscr{A}) \subseteq E^{-}(E(E^{-}(\mathscr{A})))$. Hence we have $E^{-}(\mathscr{A}) = E^{-}(E(E^{-}(\mathscr{A})))$.

    Now, consider the mapping $M \mapsto E^{-}(Z[M])$ from the set of all maximal ideals in $C(X)$ to the set of all maximal ideals in $C_b(X)$ and the mapping $M_b \mapsto Z^{\leftarrow}[\mathscr{A}]$, where $\mathscr{A}$ is the unique z-ultrafilter containing the e-ultrafilter $E(M_b)$, from the set of all maximal ideals in $C_b(X)$ to the set of all maximal ideals in $C(X)$. We will show that these mappings are inverse to each other.

    Start with any maximal ideal M in $C(X)$. Let $\mathscr{A}$ be the unique z-ultrafilter containing the e-ultrafilter $E(E^{-}(Z[M]))$. This is just $Z[M]$. Since M was a z-ideal, we have $Z^{\leftarrow}[Z[M]] = M$.

    Now, start with any maximal ideal M_b in $C_b(X)$. Let $\mathscr{A}$ be the unique z-ultrafilter containing the e-ultrafilter $E(M_b)$. Consider $E^{-}(Z[Z^{\leftarrow}[\mathscr{A}]])$. Since $\mathscr{A}$ is a z-ultrafilter, we have $Z[Z^{\leftarrow}[\mathscr{A}]] = \mathscr{A}$. Hence we have $E^{-}(Z[Z^{\leftarrow}[\mathscr{A}]]) = E^{-}(\mathscr{A})$. But now we have $E^{-}(\mathscr{A}) = E^{-}(E(^{-}(\mathscr{A})))$. Also, both $E(E^{-}(\mathscr{A}))$ and $E(M_b)$ are e-ultrafilters contained in $\mathscr{A}$, showing that $E(M_b) = E(E^{-}(\mathscr{A}))$ and hence $E^{-}(\mathscr{A}) = E^{-}(E(M_b)) = M_b$.

    This shows that the two mappings are inverse to each other and hence they are bijections between the set of all maximal ideals in $C(X)$ and the set of all maximal ideals in $C_b(X)$. □

2M The uniform norm topology on C_b

Problem

Let C' be a subring of $C(X)$ on which a topology has been defined. Then C' is called a topological ring if addition, negation and ring multiplication are continuous (from $C' \times C'$ to C' or C' to C'). If C' contains the constant functions, then it is a topological vector space if both addition and scalar multiplication (the latter being the mapping $(r, g) \mapsto \mathbb{r}g$ from $\mathbb{R} \times C'$ to C') are continuous. If C' is both a topological ring and a topological vector space, it is called a topological algebra.

By a norm is meant a mapping $f \mapsto \|f\|$ to $\mathbb{R}$, satisfying: $\|f\| \geq 0$, $\|f\| = 0$ if and only if $f = \mathbb{0}$, $\|f+g\| \leq \|f\|+\|g\|$, and $\|\mathbb{r}f\| = |r| \cdot \|f\|$. A metric d is defined from the norm, as usual, by: $d(f, g) = \|f - g\|$. A Banach algebra is a complete normed algebra whose norm satisfies: $\|fg\| \leq \|f\| \cdot \|g\|$.

1. In any topological ring, the closure of an ideal is either an ideal or the whole ring.
2. A norm on $C_b(X)$ is given by: $\|f\| = \sup_{x \in X} |f(x)|$. The resulting metric topology is called the uniform norm topology on $C_b(X)$. Convergence in this topology is uniform convergence of the functions. A base for the neighbourhood system at g consists of all sets of the form
$$\{f \mid |g - f| \leq \boldsymbol{\varepsilon}\}$$
for $\varepsilon > 0$. Equivalently, a base at g is given by all sets
$$\{f \mid |g - f| \leq u\}$$
where u is a positive unit of $C_b(X)$.
3. $C_b(X)$ is a Banach algebra.
4. The closure of every ideal is an ideal. Hence every maximal ideal is closed.
5. Every e-ideal is closed. (Hence every maximal ideal is closed.) (It will be seen subsequently (6A.2) that every closed ideal is an intersection of maximal ideals. It follows that the closed ideals are precisely the e-ideals.)
6. The topology of uniform convergence can also be defined on $C(X)$, the neighbourhood system at g being as described in (2). However, $C(X)$ will not be either a topological ring or a topological vector space unless X is pseudocompact.

Solution

1. Let I be an ideal in a topological ring R. We want to show that $\operatorname{cl} I$ is either an ideal or all of R. So, take any $x, y \in \operatorname{cl} I$. We need to show that $x + y \in \operatorname{cl} I$, meaning that for any open neighbourhood U of $x + y$, $U \cap I \neq \emptyset$. By continuity of addition, there are open neighbourhoods V of x and W of y such that $V + W \subseteq U$. But now $V \cap I \neq \emptyset$ and $W \cap I \neq \emptyset$, showing that $(V + W) \cap I \neq \emptyset$ and hence $U \cap I \neq \emptyset$. Hence $x + y \in \operatorname{cl} I$. Next, take any $x \in \operatorname{cl} I$ and any $y \in R$. We need to show that for any open neighbourhood U of xy, $U \cap I \neq \emptyset$. By continuity of multiplication, there is an open neighbourhood V of x with $V \cdot y \subseteq U$. But now $V \cap I \neq \emptyset$ and hence $(V \cdot y) \cap I \neq \emptyset$ and also $U \cap I \neq \emptyset$. This shows that $xy \in \operatorname{cl} I$. Hence $\operatorname{cl} I$ is either an ideal or all of R. □
2. The only thing that is not well-known is that the sets of the form $\{f \mid |g - f| \leq u\}$ are a neighbourhood base. This follows immediately from the fact that for every positive unit u of $C_b(X)$, there is an $\varepsilon > 0$ with $\varepsilon \leq u$ and that every $\varepsilon > 0$, considered as a constant function, is a positive unit of $C_b(X)$. □
3. It is well-known that $C_b(X)$ equipped with $\|\cdot\|$ is a Banach space. Additionally, we clearly have $\|fg\| \leq \|f\| \cdot \|g\|$, showing that $C_b(X)$ is indeed a Banach algebra. □
4. Let $I \subseteq C_b(X)$ be an ideal. Assume that $\operatorname{cl} I$ is not a proper ideal but rather the whole ring. Then we have $\mathbb{1} \in \operatorname{cl} I$. In particular, $\left\{f \mid |\mathbb{1} - f| \leq \frac{1}{2}\right\} \cap I \neq \emptyset$. But any such function is invertible.

   If $M_b \subseteq C_b(X)$ is a maximal ideal, we must have $M_b = \operatorname{cl} M_b$, showing that M_b is closed. □
5. Let $I \subseteq C_b(X)$ be an e-ideal. Take any $g \in \operatorname{cl} I$. We intend to show that $g \in E^{-}(E(I)) = I$. This means that we need to show that for any $\varepsilon > 0$, $E_\varepsilon(g) \in E(I)$. Indeed, $\left\{f \mid |g - f| \leq \frac{1}{2}\varepsilon\right\} \cap I \neq \emptyset$. Take such an f. Taking any $x \in X$ with $|f(x)| \leq \frac{1}{2}\varepsilon$, we have $|g(x)| \leq |g(x) - f(x)| + |f(x)| \leq \varepsilon$. This shows that $E_{\frac{1}{2}\varepsilon}(f) \subseteq E_\varepsilon(g)$. Due to $E_{\frac{1}{2}\varepsilon}(f) \in E(I)$, this also shows that $E_\varepsilon(g) \in E(I)$. Hence we have $g \in E^{-}(E(I)) = I$, meaning that $\operatorname{cl} I = I$.

   Since every maximal ideal is an e-ideal, this also shows that any maximal ideal of $C_b(X)$ is closed. □

6. If X is not pseudocompact, multiplication is not continuous: There is some unbounded function $f \in C(X)$. We have $\frac{1}{n} \to 0$, but $\frac{1}{n} f \not\to 0$. □

2N The m-topology on C

Problem

The m-topology is defined on $C(X)$ by taking as a base for the neighbourhood system at g all sets of the form

$$\{f \in C(X) \mid |g - f| \leq u\}$$

where u is a positive unit of $C(X)$. The same topology results if it is required further that u be a bounded function.

1. $C(X)$ is a topological ring.
2. The relative m-topology on $C_b(X)$ contains the uniform norm topology, and the two coincide if and only if X is pseudocompact. In fact, when X is not pseudocompact, the set of constant functions in $C_b(X)$ is discrete (in the m-topology), so that $C_b(X)$ is not even a topological vector space.
3. The set of all units of $C(X)$ is open, and the mapping $f \mapsto f^{-1}$ is a homeomorphism of this set onto itself.
4. The subring $C_b(X)$ is closed.
5. The closure of every ideal is an ideal. Hence every maximal ideal is closed. Every maximal ideal in $C_b(X)$ is closed.
6. Every closed ideal in $C(X)$ is a z-ideal.
7. In the ring $C(\mathbb{R})$, the z-ideal O_0 of all functions that vanish on a neighbourhood of 0 is not closed.

Solution

1. First, we will show that addition is continuous. Take any $f, g \in C(X)$ and any positive unit u of $C(X)$. Set $W = \{h \in C(X) \mid |h - (f + g)| \leq u\}$. Let $U = \left\{h \in C(X) \mid |h - f| \leq \frac{1}{2}u\right\}$, $V = \left\{h \in C(X) \mid |h - g| \leq \frac{1}{2}u\right\}$. Then we have $f \in U$, $g \in V$ and $U + V \subseteq W$. This shows that addition is a continuous map.

   Next, we will show that negation is continous. Take any $f \in C(X)$ and any positive unit u of $C(X)$. Set $V = \{g \in C(X) \mid |g + f| \leq u\}$. Let $U = \{g \in C(X) \mid |g - f| \leq u\}$. Then we have $f \in U$ and $-U \subseteq V$, showing that negation is a continuous map.

   Finally, we will show that multiplication is continuous. Take any $f, g \in C(X)$ and any positive unit u of $C(X)$. Set $W = \{h \in C(X) \mid |h - fg| \leq u\}$. Take

   $$v = \frac{1}{3}\frac{u}{|f| + 1} \wedge \frac{1}{3}\frac{u}{|g| + 1} \wedge \sqrt{\frac{1}{3}u}$$

   Take $U = \{h \in C(X) \mid |h - f| \leq v\}$ and $V = \{h \in C(X) \mid |h - g| \leq v\}$. Then we have $f \in U$, $g \in V$ and for $h \in U$, $i \in V$ and $x \in X$, we have

   $$\begin{aligned}
   & |h(x)i(x) - f(x)g(x)| \\
   = \;& |h(x)i(x) - h(x)g(x) + h(x)g(x) - f(x)g(x)| \\
   \leq \;& |h(x)i(x) - h(x)g(x)| + |h(x)g(x) - f(x)g(x)| \\
   = \;& |h(x)| \cdot |i(x) - g(x)| + |g(x)| \cdot |h(x) - f(x)| \\
   \leq \;& |h(x)| \cdot v(x) + |g(x)| \cdot v(x) \\
   = \;& |h(x) - f(x) + f(x)| \cdot v(x) + |g(x)| \cdot v(x) \\
   \leq \;& (|h(x) - f(x)| + |f(x)|) \cdot v(x) + |g(x)| \cdot v(x) \\
   = \;& |h(x) - f(x)| \cdot v(x) + |f(x)| \cdot v(x) + |g(x)| \cdot v(x) \\
   \leq \;& v(x) \cdot v(x) + |f(x)| \cdot v(x) + |g(x)| \cdot v(x) \\
   \leq \;& \sqrt{\tfrac{1}{3}u(x)} \cdot \sqrt{\tfrac{1}{3}u(x)} + |f(x)| \cdot \tfrac{1}{3}\tfrac{u(x)}{|f(x)|+1} + |g(x)| \cdot \tfrac{1}{3}\tfrac{u(x)}{|g(x)|+1} \\
   \leq \;& \tfrac{1}{3}u(x) + \tfrac{1}{3}u(x) + \tfrac{1}{3}u(x) \\
   \leq \;& u(x)
   \end{aligned}$$

   This shows that $U \cdot V \subseteq W$. Hence, multiplication is continuous.

   In total, we have shown that $C(X)$ is a topological ring. □

2. The fact that the m-topology on $C_b(X)$ contains the uniform norm topology is trivial since any positive unit u of $C_b(X)$ is also a positive unit of $C(X)$. If X is pseudocompact, these topologies clearly coincide. If X is not pseudocompact, $C(X)$ contains a positive unit u which is not bounded below, so $\{f \mid |f| \leq u\}$ is a neighbourhood of 0 in the m-topology, but not in the uniform norm topology. This shows that the m-topology and the uniform norm topology coincide if and only if X is pseudocompact.

   Now, assume that X is not pseudocompact. Let u be a positive unit in $C(X)$ which is not bounded below. Then for each $r \in \mathbb{R}$, $\{f \mid |f - \mathbb{r}| \leq u\}$ is a neighbourhood of $\mathbb{r}$ containing, of the constant functions, only $\mathbb{r}$ itself. This shows that the set of constant functions is discrete. Hence, in the m-topology, neither $C_b(X)$ nor $C(X)$ are topological vector spaces. □

3. Let $C(X)^\times$ be the set of units of $C(X)$.

   Let $f \in C(X)^\times$. Let $g = |f|$. Since f is a unit of $C(X)$, g is a positive unit of $C(X)$. Now consider the set $U = \left\{h \in C(X) \mid |f - h| \leq \frac{1}{2}g\right\}$. Then we clearly have $f \in U$ and $U \subseteq C(X)^\times$. This means that $C(X)^\times$ is a neighbourhood of each of its elements, meaning that $C(X)^\times$ is open.

   Next, consider the mapping $\cdot^{-1} : C(X)^\times \to C(X)^\times, f \mapsto f^{-1}$. Take any $f \in C(X)^\times$ and any positive unit u of $C(X)$. Consider $U = \{g \in C(X) \mid |f^{-1} - g| \leq u\}$. We intend to show that there is a neighbourhood V of f with $V^{-1} \subseteq U$. Let

$$v = \frac{1}{2}|f| \wedge \frac{1}{2}\frac{u}{f^2+1}$$

   Let $V = \{g \in C(X) \mid |f - g| \leq v\}$. We clearly have $f \in V$. Taking any $g \in V$ and $x \in X$, we have $|f(x) - g(x)| \leq \frac{1}{2}|f(x)|$, so $|g(x)| \geq \frac{1}{2}|f(x)|$ and hence

$$\begin{array}{cl} & |f^{-1}(x) - g^{-1}(x)| \\ = & \left|\frac{1}{f(x)} - \frac{1}{g(x)}\right| \\ = & \left|\frac{g(x)-f(x)}{f(x)g(x)}\right| \\ \leq & \left|\frac{g(x)-f(x)}{f(x)\frac{1}{2}f(x)}\right| \\ = & 2\left|\frac{g(x)-f(x)}{f(x)^2}\right| \\ = & 2\frac{|g(x)-f(x)|}{f(x)^2} \\ \leq & 2\frac{\frac{1}{2}\frac{u(x)}{f(x)^2+1}}{f(x)^2} \\ \leq & u(x) \end{array}$$

   This shows that $|f^{-1} - g^{-1}| \leq u$, so $g^{-1} \in U$ and hence $V^{-1} \subseteq U$. Hence $\cdot^{-1}$ is continuous. Since it is its own inverse, it is even a homeomorphism. □

4. Let $U = C(X) \setminus C_b(X)$. We will show that U is open, showing that $C_b(X)$ is closed. Take any $f \in U$. Set $u = \frac{1}{2}|f|$ and let $V = \{g \in C(X) \mid |g - f| \leq u\}$. Since f is unbounded, all elements of V are unbounded. Hence around any unbounded function $f \in U$, there is a neighbourhood V of unbounded functions. This shows that U is open and hence that $C_b(X)$ is closed. □

5. Let $I \subseteq C(X)$ be an ideal. Assume that $\operatorname{cl} I$ is not a proper ideal but rather the whole ring. Then we have $\mathbb{1} \in \operatorname{cl} I$. In particular, $\left\{f \mid |\mathbb{1} - f| \leq \frac{1}{2}\right\} \cap I \neq \emptyset$. But any such function is invertible.

   If $M \subseteq C(X)$ is a maximal ideal, we must have $M = \operatorname{cl} M$, showing that M is closed.

   If $M_b \subseteq C_b(X)$ is a maximal ideal, we must have $M_b = \operatorname{cl} M_b$, showing that M_b is closed. □

6. Let I be a closed ideal in $C(X)$. Take $Z(f) \in Z[I]$. Let $g \in C(X)$ with $Z(g) = Z(f)$. We need to show that $g \in I$. Since I is closed, it suffices to show that $g \in \operatorname{cl} I$. So, take any positive unit u of $C(X)$. Let $U = \{h \in C(X) \mid |g - h| \leq u\}$. We must show that $U \cap I \neq \emptyset$. Indeed, define

$$h : X \to \mathbb{R}, h(x) = \begin{cases} 0 & |g(x)| \leq u(x) \\ \frac{g(x) \pm u(x)}{f(x)} & \text{otherwise} \end{cases}$$

   Clearly, $hf \in U$ and due to $f \in I$ also $hf \in I$. This shows that $U \cap I \neq \emptyset$. Hence $g \in \operatorname{cl} I = I$, so I is indeed a z-ideal. □

7. We wish to show that O_0 is not a closed ideal in $C(\mathbb{R})$. Indeed, $\mathbb{0} \notin O_0$. However, for any positive unit u of $C(X)$, consider $U = \{f \in C(\mathbb{R}) \mid |f - \mathbb{0}| \leq u\}$. Since u is a positive unit, around 0, it is bounded away from 0. This means that we can find some $f \in U$ which is zero on some neighbourhood of 0. But then also $f \in O_0$, showing that $U \cap O_0 \neq \emptyset$ and hence $\mathbb{0} \in \operatorname{cl} O_0$. This shows that O_0 is not a closed ideal in $C(\mathbb{R})$. □

3 Completely regular spaces

3A Zero-divisors, units, square roots

Problem

Let X be a completely regular space containing more than one point.

1. $C_b(X)$, and hence $C(X)$, contains zero divisors (i.e., it is not an integral domain).
2. $C_b(X)$, and hence $C(X)$, contains nonconstant units
3. Let $\mathfrak{m}$ be an infinite cardinal, and let X be the one-point compactification of the discrete space of power $\mathfrak{m}$. In $C(X)$, $\mathbb{1}$ has just $\mathfrak{m}$ square roots.

Solution

1. Let $x \neq y$ be distinct points of X. Since X is completely regular, x and y are completely separated via the continuous function f, i.e. $f(x) = 0$ and $f(y) = 1$. Now consider the functions

$$g : \mathbb{R} \to \mathbb{R}, g(x) = \begin{cases} 0 & x \leq \frac{1}{2} \\ 2x - 1 & \frac{1}{2} \leq x \leq 1 \\ 1 & 1 \leq x \end{cases}$$

and

$$h : \mathbb{R} \to \mathbb{R}, h(x) = \begin{cases} 1 & x \leq 0 \\ 1 - 2x & 0 \leq x \leq \frac{1}{2} \\ 0 & \frac{1}{2} \leq x \end{cases}$$

Then we have $g \circ f \in C_b(X)$, $h \circ f \in C_b(X)$, $(g \circ f)(y) = g(f(y)) = g(1) = 1$, $(h \circ f)(x) = h(f(x)) = h(0) = 1$ and for any $z \in X$: $(g \circ f)(x) \cdot (h \circ f)(x) = ((g \cdot h) \circ f)(x) = 0$. This shows that $g \circ f$ and $h \circ f$ are nonzero elements in $C_b(X)$ whose product is zero. Hence $C_b(X)$ contains zero divisors. □

2. Take distinct points $x \neq y$ in X. They are completely separated by some continuous function f with $f(x) = 0$ and $f(y) = 1$ taking values only in $[0, 1]$. Then $f + 1$ is a nonconstant unit of $C_b(X)$. □

3. The square roots of $\mathbb{1}$ are precisely the functions taking only 1 and -1 as values. Consider some $f \in C(\mathfrak{m}^*)$ with $f^2 = \mathbb{1}$. W.l.o.g. assume that $f(\infty) = 1$. Due to continuity of f, there must be some neighbourhood of ∞ on which f is constant, but the neighbourhoods of ∞ are precisely the complements of finite subsets of $\mathfrak{m}$. Thus, f can only take the value -1 on some finite subset of $\mathfrak{m}$. Hence, there are as many square roots of $\mathbb{1}$ as there are finite subsets of $\mathfrak{m}$ and since $\mathfrak{m}$ is infinite, there are $\mathfrak{m}$ finite subsets of $\mathfrak{m}$. This shows that in $C(\mathfrak{m}^*)$, $\mathbb{1}$ has $\mathfrak{m}$ square roots. □

3B Countable sets

Problem

Let X be a completely regular space.

1. A countable set disjoint from a closed set F is disjoint from some zero-set containing F.
2. A C-embedded countable set S is completely separated from every disjoint closed set. (This is false if S is uncountable (5.13) or if S is only C_b-embedded (4M), even if S is closed (8.20 and 6P).)
3. Any C-embedded countable set is closed. (An uncountable C-embedded set need not be closed (5.13); the appropriate generalization is in 8A.1.)
4. Any two countable sets, neither of which meets the closure of the other, are contained in disjoint cozero-sets. (But the given sets need not be completely separated, even if they are closed; see 8J.4.)
5. A countable, completely regular space is normal. (More generally, see 3D.4.)

Solution

1. Let $A = \{a_n\}_{n \in \mathbb{N}} \subseteq X$ be countable and $F \subseteq X$ closed and disjoint from A. Now, for each $n \in \mathbb{N}$, $a_n \notin F$. Since X is completely regular, this implies that there exists a zero-set Z_n with $a_n \notin Z_n$ and $Z_n \supseteq F$. Let $Z = \bigcap_{n \in \mathbb{N}} Z_n$. Then Z is a zero-set with $A \cap Z = \emptyset$ and $Z \supseteq F$. □

2. Let $S \subseteq X$ be countable and C-embedded and let $F \subseteq X$ be a closed set disjoint from S. By (1), there exists a zero-set Z with $S \cap Z = \emptyset$ and $Z \supseteq F$. By Theorem 1.18, S is completely separated from Z and hence also completely separated from F. □

3. Let $S \subseteq X$ be countable and C-embedded. Take any $x \notin S$. Then $\{x\}$ is a closed set disjoint from S, so by (2), S and x are completely separated. In particular, there is a zero-set-neighbourhood Z of x disjoint from S. This shows that any point not in S has a neighbourhood disjoint from S. Hence the complement of S is open and S itself is closed. □

4. Take countable sets $A = \{a_n\}_{n\in\mathbb{N}}$ and $B = \{b_n\}_{n\in\mathbb{N}}$ such that $\operatorname{cl} A \cap B = \emptyset$ and $A \cap \operatorname{cl} B = \emptyset$.

   The point a_1 is completely separated from the closed set $\operatorname{cl} B$. Hence there is a continuous function f_1 with $f_1(a_1) = 0$ and $f_1[\operatorname{cl} B] \subseteq \{1\}$ taking values only in $[0, 1]$. Now let $C_1 = f_1^{\leftarrow}\left[[0, \frac{1}{2})\right]$ and $\tilde{C}_1 = f_1^{\leftarrow}\left[[0, \frac{1}{2}]\right]$. Next, b_1 is completely separated from $\operatorname{cl} A \cup \tilde{C}_1$ via a continuous function g_1 with $g_1(b_1) = 0$ and $g_1[\operatorname{cl} A \cup \tilde{C}_1] \subseteq \{1\}$ taking values only in $[0, 1]$. Let $D_1 = g_1^{\leftarrow}\left[[0, \frac{1}{2})\right]$ and $\tilde{D}_1 = g_1^{\leftarrow}\left[[0, \frac{1}{2}]\right]$.

   We will continue in this fashion. a_2 and $\operatorname{cl} B \cup \tilde{D}_1$ are completely separated via f_2, next set $C_2 = f_2^{\leftarrow}\left[[0, \frac{1}{2})\right]$ and $\tilde{C}_2 = f_2^{\leftarrow}\left[[0, \frac{1}{2}]\right]$. b_2 and $\operatorname{cl} A \cup \tilde{C}_1 \cup \tilde{C}_2$ are completely separated via g_2, next set $D_2 = g_2^{\leftarrow}\left[[0, \frac{1}{2})\right]$ and $\tilde{D}_2 = g_2^{\leftarrow}\left[[0, \frac{1}{2}]\right]$.

   ...

   In the end, $\bigcup_{n\in\mathbb{N}} C_n$ and $\bigcup_{n\in\mathbb{N}} D_n$ will be disjoint cozero-sets containing A and B, respectively. □

5. If X is countable and completely regular, taking any two closed subsets, by (4) they are actually contained in disjoint open sets, showing that X is normal. □

3C G_δ-points of a completely regular space

Problem

Let p be a G_δ-point of a completely regular space X, and let $S = X \setminus \{p\}$.

1. If $g \in C_b(S)$, $h \in C(X)$, and $h(p) = 0$, then $g \cdot h|_S$ has a continuous extension to all of X.
2. If Z is a zero-set in S, then $\operatorname{cl}_X Z$ is a zero-set in X.

Solution

1. Define

$$f : X \to \mathbb{R}, f(x) = \begin{cases} 0 & x = p \\ g(x) \cdot h(x) & x \in S \end{cases}$$

   Since there is some constant M with $|g(x)| \leq M$, we have $\lim_{x\to p} |f(x)| \leq M \lim_{x\to p} |h(x)| = 0$ and hence f is continuous at p. □

2. Consider $Z = Z(f)$ for $f \in C(S)$ where f takes values in $[0, 1]$. Since p is a G_δ-point, it is in fact a zero-set, so we can write $\{p\} = Z(g)$ for some $g \in C(X)$ taking values in $[0, 1]$.

   Now, if $p \in \operatorname{cl}_X Z$, let $h = f$. Then by (1), $h \cdot g|_S$ has a continuous extension to all of X, and the zero-set of this extension will simply be $Z \cup \{p\} = \operatorname{cl}_X Z$.

   On the other hand, if $p \notin \operatorname{cl}_X Z$, p and Z must in fact be completely separated in X, so that we may assume $g[Z] \subseteq \{1\}$. Then we can define $h = \mathbb{1} - f$. Again, $h \cdot g|_S$ has a continuous extension to all of X. The set where this extension attains the value 1 is precisely $Z = \operatorname{cl}_X Z$. □

3D Normal spaces

Problem

1. The following are equivalent for any Hausdorff space X.

   (1) X is normal.

   (2) Any two disjoint closed sets are completely separated.

   (3) Every closed set is C_b-embedded.

   (4) Every closed set is C-embedded.

2. Every normal pseudocompact space is countably compact. (But a nonnormal, pseudocompact space need not be countably compact; see 5I or 8.20. And even a normal, pseudocompact space need not be compact; see 5.12.)

3. Every closed G_δ in a normal space is a zero-set.

4. Every completely regular space with the Lindelöf property is normal. (It is well known, more generally, that every regular Lindelöf space is normal.)

5. Let X be a completely regular space. If $X = S \cup K$, where S is open and normal, and K is compact, then X is normal.

Solution

1. (1) $\implies$ (2) This is just Urysohn's Lemma 3.13.

   (2) $\implies$ (3) If $A \subseteq X$ is closed and $B, C \subseteq A$ are completely separated in A, then their closures in A are disjoint. Since A is closed in X, their closures in X are also disjoint and hence they are completely separated. Urysohn's extension Theorem 1.17 now implies that A is C_b-embedded in X.

   (3) $\implies$ (4) Let $A \subseteq X$ be a closed set. It is C_b-embedded in X. Now take any zero-set Z which is disjoint from A. Then $A \cup Z$ is a closed set and hence C_b-embedded in X. Hence the indicator function of A on $A \cup Z$ has a continuous extension to all of X, showing that A and Z are completely separated in X. Theorem 1.18 now implies that A is even C-embedded.

   (4) $\implies$ (1) Let $A, B \subseteq X$ be disjoint open sets. Then $A \cup B$ is also a closed set in X and is hence C-embedded in X. Then the indicator function of A, which is continuous since A is clopen in $A \cup B$, has a continuous extension to all of X. Hence A and B are completely separated in X and in particular contained in disjoint neighbourhoods. □

2. Let X be a normal pseudocompact space. Striving for a contradiction, assume that X is not countably compact. This means that there is a countable open cover $\{U_n\}_{n\in\mathbb{N}}$ without a finite subcover. W.l.o.g. assume that $U_1 \subsetneq U_2 \subsetneq U_3 \subsetneq \ldots$. Pick $x_2 \in U_2 \setminus U_1$, $x_3 \in U_3 \setminus U_2$, $x_4 \in U_4 \setminus U_4$, $\ldots$. We intend to show that $\{x_n\}_{n\in\mathbb{N}_{\geq 2}}$ has no limit points. Striving for a contradiction, assume that $p \in X$ was a limit point of $\{x_n\}_{n\in\mathbb{N}_{\geq 2}}$. Since $\{U_n\}_{n\in\mathbb{N}}$ is a cover of X, we must have some $n \in \mathbb{N}$ with $p \in U_n$. Since X is Hausdorff, we can find an open neighbourhood V of p such that $x_2, \ldots, x_n \notin V$. But then $U_n \cap V$ is an open neighbourhood of p disjoint from $\{x_n\}_{n\in\mathbb{N}_{\geq 2}}$, a contradiction. Hence $\{x_n\}_{n\in\mathbb{N}_{\geq 2}}$ has no limit points. In particular, it is closed and discrete. But now $\{x_n\}_{n\in\mathbb{N}_{\geq 2}}$ is a closed set in a normal space and hence by (1) it is C-embedded. A pseudocompact space cannot contain an infinite discrete C-embedded subset. This contradiction shows that X must have been countably compact. □

3. Let X be normal. Let A be a closed G_δ with $A = \bigcap_{n\in\mathbb{N}} U_n$ where each U_n is open. Then A is completely separated from each $X \setminus U_n$, so there exist zero-sets Z_n with $A \subseteq Z_n$ and $Z_n \cap (X \setminus U_n) = \emptyset$, i.e. $Z_n \subseteq U_n$. But then we have $A \subseteq \bigcap_{n\in\mathbb{N}} Z_n \subseteq \bigcup_{n\in\mathbb{N}} U_n = A$, showing that $A = \bigcap_{n\in\mathbb{N}} Z_n$ and hence A is also a zero-set. □

4. Let X be completely regular and Lindelöf. We will show that given any two closed sets A and B, there exists a zero-set Z with $B \subseteq Z$ and $A \cap Z = \emptyset$. Then, given any two closed sets A and B, we will be able to apply this in order to obtain first a zero-set Z_1 with $B \subseteq Z_1$ and $A \cap Z_1 = \emptyset$ and then a zero-set Z_2 with $A \subseteq Z_2$ and $Z_2 \cap Z_1 = \emptyset$. This shows that any two disjoint closed sets A and B are contained in disjoint zero-sets and hence are completely separated.

   So, let A and B be disjoint closed sets in X. For each $x \in A$, since X is completely regular, there exists a zero-set Z_x with $x \notin Z_x$ and $B \subseteq Z_x$. Then $\{X \setminus Z_x\}_{x\in A}$ is an open cover of A. Since A is a closed subspace of a Lindelöf space, it is itself Lindelöf. Hence there is a countable collection $\{x_n\}_{n\in\mathbb{N}}$ such that $\{X \setminus Z_{x_n}\}_{n\in\mathbb{N}}$ is still a cover of A. But now $Z = \bigcap_{n\in\mathbb{N}} Z_n$ is a zero-set containing B and disjoint from A. Hence for any two disjoint closed sets A and B, there is a zero-set Z containing B and disjoint from A. □

5. Let X be completely regular and assume that it can be written as $X = S \cup K$ with S open and normal and K compact. Note that we do not assume S and K to be disjoint.

   Let $A, B \subseteq X$ be two disjoint closed sets.

   We know that $A \cap K$ is a compact set disjoint from the closed set B in X. Since X is completely regular, this implies that there are open sets O and P in X with $A \cap K \subseteq O$, $B \subseteq P$ and $O \cap P = \emptyset$. Analogously, $B \cap K$ is a compact set disjoint from the closed set A in X, which by complete regularity of X implies the existence of open sets Q and R in X with $B \cap K \subseteq Q$, $A \subseteq R$ and $Q \cap R = \emptyset$.

Next, $A \cap S$ and $B \cap S$ are disjoint closed sets in the normal space S and hence there must be open sets T and U in S with $A \cap S \subseteq T$, $B \cap S \subseteq U$ and $T \cap U = \emptyset$. Since S is open in X, T and U are even open in X.

Now we have $A \subseteq R$ and

$$A = A \cap X = A \cap (S \cup K) = (A \cap S) \cup (A \cap K) \subseteq T \cup O$$

showing that $A \subseteq R \cap (T \cup O)$. This means that $R \cap (T \cup O)$ is an open neighbourhood of A.

We also have $B \subseteq P$ and

$$B = B \cap X = B \cap (S \cup K) = (B \cap S) \cup (B \cap K) \subseteq U \cup Q$$

showing that $B \subseteq P \cap (U \cup Q)$. This means that $P \cap (U \cup Q)$ is an open neighbourhood of A.

Now take any point $x \in [R \cap (T \cup O)] \cap [P \cap (U \cup Q)]$. Since $x \in T \cup O$, we either have $x \in T$ or $x \in O$. Since $x \in U \cup Q$, we either have $x \in U$ or $x \in Q$. We cannot have $x \in O$, since then we would have $x \in O \cap P = \emptyset$, a contradiction. We also cannot have $x \in Q$, since then we would have $x \in Q \cap R = \emptyset$, a contradiction. Hence we must have $x \in T$ and $x \in U$, showing that $x \in T \cap U = \emptyset$, a contradiction. This contradiction shows that $[R \cap (T \cup O)] \cap [P \cap (U \cup Q)] = \emptyset$.

But now $(O \cup T) \cap R$ and $(Q \cup U) \cap P$ are disjoint open neighbourhoods of A and B in X, showing that X is normal. □

3E Nonnormal spaces

Problem

Let X be a nonnormal, Hausdorff space.

1. X contains a closed set that is not a zero-set.
2. X has a subspace S with the following property: any two completely separated sets in S have disjoint closures in X, yet S is not C_b-embedded in X. Compare Urysohn's extension theorem.

Solution

1. By 3D.2, X contains two disjoint closed sets which are not completely separated. Hence by Theorem 1.17, they cannot both be zero-sets. □
2. By 3D.3, X, contains a closed subset S which is not C_b-embedded. However, any two completely separated sets in S have disjoint closures in S and hence also in X. □

3F T_0-spaces

Problem

In a topological space X, define $x \equiv x'$ to mean that $\mathrm{cl}\{x\} = \mathrm{cl}\{x'\}$. Let Y be the set of all equivalence classes thus defined, let τ map each point of X into its equivalence class, and provide Y with the quotient topology relative to τ.

1. For E closed in X, if $x \in E$ and $x \equiv x'$, then $x' \in E$; hence $\tau[X \setminus E] = Y \setminus \tau[E]$.
2. The weak topology induced by τ agrees with the given topology on X.
3. τ is both an open mapping and a closed mapping.
4. $\tau[\mathrm{cl}\{x\}] = \mathrm{cl}\{\tau(x)\}$, whence Y is a T_0-space.
5. In Theorem 3.2 (and hence in Theorems 3.6 and 3.7), it is enough to require that X be a T_0-space, rather than a Hausdorff space.

Solution

1. Let E be closed in X.

   Assume that $x \in E$ and $x \equiv x'$, meaning that $\mathrm{cl}\{x\} = \mathrm{cl}\{x'\}$. Since E is closed, we have $x \in \mathrm{cl}\{x\} \subseteq E$ and hence also $\mathrm{cl}\{x'\} \subseteq E$, implying that $x' \in E$.

   Now, we will show that $\tau[X \setminus E] = Y \setminus \tau[E]$.

Take some $x \in X \setminus E$. Striving for a contradiction, assume that $\tau(x) \in \tau[E]$. This means that there is some $x' \in E$ with $\tau(x) = \tau(x')$, meaning that $x \equiv x'$. By the above, this would imply that $x \in E$, a contradiction. This contradiction shows that $\tau(x) \in Y \setminus \tau[E]$. We have shown that $\tau[X \setminus E] \subseteq Y \setminus \tau[E]$.

On the other hand, take any $y \in Y \setminus \tau[E]$. Since τ is surjective, there must be some $x \in X$ with $y = \tau(x)$. Since $y \notin \tau[E]$, we have $x \notin E$, hence $x \in X \setminus E$, showing that $y \in \tau[X \setminus E]$. This shows that $Y \setminus \tau[E] \subseteq \tau[X \setminus E]$.

In total, we have shown that $\tau[X \setminus E] \subseteq Y \setminus \tau[E]$ and that $\tau[X \setminus E] \supseteq Y \setminus \tau[E]$, implying that $\tau[X \setminus E] = Y \setminus \tau[E]$. □

2. Take a subbasic open set U of the weak topology on X, which is of the form $\tau^{\leftarrow}[V]$ for some open set V in Y. By definition of the quotient topology, V is open if and only if $\tau^{\leftarrow}[V]$ is open in X with the given topology. This shows that any subbasic open set of the weak topology is also open in the given topology. Hence every open set of the weak topology is also open in the given topology. This shows that the weak topology is coarser than the given topology.

   Let E be any closed set in the given topology on X. We will show that $\tau^{\leftarrow}[\tau[E]] = E$. Indeed, it is clear that $\tau^{\leftarrow}[\tau[E]] \supseteq E$. Now take any $x \in \tau^{\leftarrow}[\tau[E]]$, meaning that $\tau(x) \in \tau[E]$. Then there is some $x' \in E$ with $\tau(x) = \tau(x')$, but by (1), this implies that $x \in E$. Hence we have $\tau^{\leftarrow}[\tau[E]] \subseteq E$, showing that $\tau^{\leftarrow}[\tau[E]] = E$.

   Now take any closed set E in the given topology on X. We need to show that E is closed in the weak topology on X, for which it suffices to show that it is the preimage of a closed set in Y. Indeed, we can consider $\tau[E]$. We have $\tau^{\leftarrow}[\tau[E]] = E$ so since E is closed in the given topology X, we have that $\tau[E]$ is closed in the quotient topology on Y. But now E is the preimage of the closed set $\tau[E]$ under τ, so it is indeed closed in the weak topology on X. This shows that the weak topology is finer than the given topology.

   In total, we see that the weak topology on X and the given topology on X must agree. □

3. Let E be closed in X. Then by the proof of (2), we have $\tau^{\leftarrow}[\tau[E]] = E$, meaning that the preimage of $\tau[E]$ under τ is a closed set in X. This shows that $\tau[E]$ is closed in the quotient topology on Y and hence τ is a closed mapping.

   Let U be open in X. Then we can write it as $U = X \setminus E$ for some closed set E in X. But then due to (1) we have $\tau[U] = \tau[X \setminus E] = Y \setminus \tau[E]$ and since τ is a closed mapping, we see that $Y \setminus \tau[E]$ is an open set in Y and hence τ is also an open mapping. □

4. Since τ is a closed mapping, $\tau[\mathrm{cl}\{x\}]$ is a closed set containing $\tau(x)$. Take any other closed set F in Y with $\tau(x) \in F$. Since F is closed in Y, $\tau^{\leftarrow}[F]$ is closed in X. We also have $x \in \tau^{\leftarrow}[F]$, showing that $\mathrm{cl}\{x\} \subseteq \tau^{\leftarrow}[F]$. This yields $\tau[\mathrm{cl}\{x\}] \subseteq \tau[\tau^{\leftarrow}[F]] = F$, where the last equality holds since τ is surjective. Hence for any closed set F in Y with $\tau(x) \in F$, we in fact have $F \supseteq \tau[\mathrm{cl}\{x\}]$, showing that $\tau[\mathrm{cl}\{x\}]$ is the smallest closed set containing $\tau(x)$ and hence $\tau[\mathrm{cl}\{x\}] = \mathrm{cl}\{\tau(x)\}$.

   Let us show that for closed set $E_1, E_2 \subseteq x$, $\tau[E_1] = \tau[E_2]$ implies $E_1 = E_2$. Indeed, take any $x \in E_1$. Then due to $\tau[E_1] = \tau[E_2]$, we have some $x' \in E_2$ with $\tau(x) = \tau(x')$. By (1), this implies that $x \in E_1$. Hence $E_1 \subseteq E_2$. We can analogously show that $E_1 \supseteq E_2$ and hence $E_1 = E_2$.

   Next, we wish to see that Y is indeed T_0, meaning that distinct points have distinct closures. Indeed, take two distinct points $\tau(x)$ and $\tau(x')$ in Y. Then by definition we have $\mathrm{cl}\{x\} \neq \mathrm{cl}\{x'\}$. But now we have $\mathrm{cl}\{\tau(x)\} = \tau[\mathrm{cl}\{x\}] \neq \tau[\mathrm{cl}\{x'\}] = \mathrm{cl}\{\tau(x')\}$. Hence distinct points in Y have distinct closures, showing that Y is T_0. □

5. Regarding Theorem 3.2, consider the following properties of an arbitrary topological space X:

   (P1) If $F \subseteq X$ is closed and $x \notin F$, then x and F are completely separated.

   (P2) The zero-sets $Z(X)$ form a base for the closed sets in X.

   Then we can show the following statements:

   (1) X is T_0 and X fulfills (P1) $\implies$ X is Hausdorff, X fulfills (P1) and X fulfills (P2)

   (2) X is T_0 and X fulfills (P2) $\implies$ X is Hausdorff, X fulfills (P1) and X fulfills (P2)

   Here are the proofs:

   (1) Assume that X is T_0 and if $F \subseteq X$ is closed and $x \notin F$, then x and F are completely separated. We intend to show that X is Hausdorff and that the zero-sets $Z(X)$ form a base for the closed sets in X.

Indeed, take distinct points x and y. Since X is T_0, w.l.o.g. $x \notin \text{cl}\{y\}$. Then x and $\text{cl}\{y\}$ must be completely separated, showing that X is Hausdorff.

Now if F is any closed set and $x \notin F$, then x and F are completely separated, meaning that there is a zero-set Z with $Z \supseteq F$ and $x \notin Z$. This shows that the zero-sets $Z(X)$ form a base for the closed sets in X.

(2) Assume that X is T_0 and that the zero-sets $Z(X)$ form a base for the closed sets in X. We intend to show that X is Hausdorff and that if $F \subseteq X$ is closed and $x \notin F$, then x and F are completely separated.

Indeed, take distinct points x and y. Since X is T_0, w.l.o.g. $x \notin \text{cl}\{y\}$. Then there must be a zero-set Z with $Z \supseteq \text{cl}\{y\}$ and $x \notin Z$. Let $Z = Z(f)$ and consider $g = \frac{1}{f(x)} \cdot f$. Then g completely separates x and $\text{cl}\{y\}$, showing that X is Hausdorff.

Now if F is a closed set and $x \notin F$, then there must be a zero-set Z with $Z \supseteq F$ and $x \notin Z$. Again, writing $Z = Z(f)$ and defining $g = \frac{1}{f(x)} \cdot f$, we see that x and F are completely separated. □

3G Weak topology

Problem

If a family C' of real-valued functions on X is an additive group, contains the constant functions, and contains the absolute value of each of its members, then the collection of sets of the form 3.4(a) is a base of neighbourhoods of x, for each $x \in X$, in the weak topology induced by C'.

Solution

Let X be a topological space, C' be a family of real-valued functions on X and assume that C' is an additive group, contains the constant functions, and contains the absolute value of each of its members. We intend to show that the collection of sets of the form

$$\{y \in X \mid |f(x) - f(y)| < \varepsilon\}$$

where $f \in C'$ and $\varepsilon > 0$ constitutes a base of neighbourhoods of x in the weak topology induced by C'. Fix $x \in X$.

First of all, we will show that every set of the above form is in fact of the form

$$\{y \in X \mid g(y) < \varepsilon\}$$

for some $g \in C'$ with $g \geq 0$ and $g(x) = 0$. For this, consider the function $g(y) = |f(x) - f(y)|$. Then we have $g(y) < \varepsilon$ if and only if we have $|f(x) - f(y)| < \varepsilon$. Additionally, since C' is an additive group, contains the constant functions, and contains the absolute value of each of its member, we have $g \in C'$.

Conversely, if we have any set of the above form for some $g \in C'$ with $g \geq 0$ and $g(x) = 0$, then it is clear that this is the same as

$$\{y \in X \mid |g(x) - g(y)| < \varepsilon\}$$

which is an element of our subbase.

It remains to show that the set of functions of the form

$$\{y \in X \mid g(y) < \varepsilon\}$$

for $g \in C'$ with $g \geq 0$ and $g(x) = 0$ constitute a base.

So, take $f, g \in C'$ with $f, g \geq 0$ and $f(x), g(x) = 0$. Additionally, take $\varepsilon, \delta > 0$. We intend to show that there is $h \in C'$ with $h \geq 0$ and $h(x) = 0$, and $\gamma > 0$ such that

$$\{y \in X \mid h(y) < \gamma\} \subseteq \{y \in X \mid f(y) < \varepsilon\} \cap \{y \in X \mid g(y) < \delta\}$$

For this, take $h = f + g$ and $\gamma = \min(\varepsilon, \delta)$. Then we clearly have $h \geq 0$, $h(x) = 0$ and $\gamma > 0$. Additionally, $h(y) < \gamma$ implies $f(x) \leq h(x) < \gamma \leq \varepsilon$ and $g(x) \leq h(x) < \gamma \leq \delta$. This shows that

$$\{y \in X \mid h(y) < \gamma\} \subseteq \{y \in X \mid f(y) < \varepsilon\} \cap \{y \in X \mid g(y) < \delta\}$$

Hence the collection of sets of the above form constitutes not just a subbase for the neighbourhoods at x, but actually a base for the neighbourhoods at x, in the weak topology induced by C'. □

3H Completely regular family

Problem

Let X be a completely regular space. A subfamily C' of $C(X)$ is called a completely regular family if whenever F is closed and $x \notin F$, there exists $f \in C'$ such that $f(x) \notin \operatorname{cl} f[F]$. For example, $\{\mathbf{i}\}$ is a completely regular family in $C(\mathbb{R})$. Every completely regular family determines the topology of X; in fact, C' is a completely regular family if and only if the collection of all sets of the form 3.4(a) $(x \in X)$ is a base for the topology.

Solution

First of all, let's show that C' is a completely regular family if and only if the collection of all sets of the form 3.4(a) $(x \in X)$ is a base for the given topology on X.

Assume that C' is a completely regular family. Take any open set U in X and any point $x \in U$. We need to show that there is a set B of the form 3.4(a) with $x \in B \subseteq U$. Indeed, consider the closed set $F = X \setminus U$. Since C' is a completely regular family, there is some $f \in C'$ such that $f(x) \notin \operatorname{cl} f[F]$. In particular, there is some $\varepsilon > 0$ with

$$\{z \in \mathbb{R} \mid |f(x) - z| < \varepsilon\} \cap f[F] = \emptyset$$

But this means that

$$\{y \in X \mid |f(x) - f(y)| < \varepsilon\} \cap F = \emptyset$$

and hence

$$\{y \in X \mid |f(x) - f(y)| < \varepsilon\} \subseteq U$$

For the converse, assume that the collection of all sets of the form 3.4(a) $(x \in X)$ is a base for the topology. Assume that F is closed in X and $x \notin F$. We need to show that there is $f \in C'$ with $f(x) \notin \operatorname{cl} f[F]$. Indeed, consider the open set $U = X \setminus F$. Since the collection of all sets of the form 3.4(a) is a base for the topology, there is some $f \in C'$ and $\varepsilon > 0$ with

$$\{y \in X \mid |f(x) - f(y)| < \varepsilon\} \subseteq U$$

and hence

$$\{y \in X \mid |f(x) - f(y)| < \varepsilon\} \cap F = \emptyset$$

But this implies that

$$\{z \in \mathbb{R} \mid |f(x) - z| < \varepsilon\} \cap f[F] = \emptyset$$

showing that $f(x) \notin \operatorname{cl} f[F]$.

Next, let's see that every completely regular family determines the topology of X. Taking a completely regular family C', we need to show that the weak topology induced by C' coincides with the given topology on X. Clearly the weak topology induced by C' is coarser than the given topology on X. By the above, there is a collection of open sets in the weak topology induced by C' which is a base of the given topology on X. This means that any set that is open in the given topology on X is also open in the weak topology induced by C'. This shows that the given topology on X coincides with the weak topology induced by C', showing that C' determines the given topology on X. □

3I Theorem 3.9

Problem

1. Let A be a nonempty subfamily of $C(X)$. Define Y and τ as in 3.9, except that now f ranges only over A, instead of over all of $C(X)$. Then Y is completely regular, τ is continuous, and the mapping $g \mapsto g \circ \tau$ is an isomorphism from $C(Y)$ into $C(X)$.
2. Let X be the set of real numbers, under the discrete topology, and take $A = \{\sigma\}$, where σ is the identity map of X onto $\mathbb{R}$. Then $\tau = \sigma$, and $Y = \mathbb{R}$. Hence τ is not a quotient mapping.

Solution

1. We define an equivalence relation on X via $x \equiv x'$ if and only if $f(x) \equiv f(x')$ for every $f \in A$. Define Y as the set of equivalence classes of X under this equivalence relation and let $\tau : X \to Y, \tau(x) = [x]$. Every $f \in A$ factors through Y, meaning that there exists a unique $g : Y \to \mathbb{R}$ with $f = g \circ \tau$. Endow Y with the weak topology with respect to the collection C' of all such g, that is, with the collection of all $g : Y \to \mathbb{R}$ such that $g \circ \tau \in A$.

Take distinct points $y \neq y'$ in Y. By construction of Y, there is some $g \in C'$ with $g(y) \neq g(y')$. This shows that Y is Hausdorff. Since its topology is additionally determined by the family of real-valued functions C', by Theorem 3.7, Y is completely regular.

By definition of the weak topology, τ is continuous if and only if every composite $g \circ \tau$ is continuous for every $g \in C'$. But this is true by definition of C'. Hence τ is continuous.

Clearly, the map $C(Y) \to C(X), g \mapsto g \circ \tau$ is a homomorphism. Now take $g \neq g'$ in $C(Y)$. In particular, there is some $y \in Y$ with $g(y) \neq g'(y)$. Since τ is surjective, there must be $x \in X$ with $\tau(x) = y$ and hence $g(\tau(x)) \neq g'(\tau(x))$, showing that $g \circ \tau \neq g' \circ \tau$. Hence $g \mapsto g \circ \tau$ is injective. $\square$

2. Clearly, $\sigma(x) = \sigma(x')$ if and only if $x = x'$ and hence the set of equivalence classes is $\mathbb{R}$, endowed with the weak topology induced by the identity function to $\mathbb{R}$. This shows that $Y = \mathbb{R}$ and $\tau = \sigma$. If τ was a quotient mapping, it would have to be a homeomorphism. Hence τ is not a quotient mapping. $\square$